Problem Statement

How much computational time does it take to conduct back substitution in the LU Decomposition method?

Solution

At the beginning of back substitution, the set of equations is of the form

$$[U][X]=[B]$$

where

$[U]=$ upper triangular matrix, $n\times n$,
$[X]=$ unknown vector, $n\times 1$, and
$[B]=$ right hand side vector, $n\times 1$.

The algorithm for finding the right hand vector is

$$x_n=\frac{b_n}{u_{nn}} \tag{1}$$

$$x_i=\frac{b_i-\sum_{j=i+1}^{n}u_{ij}x_j}{u_{ii}},\ i=n-1,n-2,\ldots,2,1 \tag{2}$$

Now let us see how many arithmetic operations are needed to complete the algorithm.

We have one operation of division for the calculation of x_n (See equation 1).

For other x_i, the arithmetic operations are as follows. For a particular i, there are $(n-i)$ multiplications – just count the $u_{ij}x_j$ terms within the summation in the numerator of equation (2). Each of these multiplication values are subtracted from a variable with an initial value of b_i. Hence, there are $(n-i)$ subtractions. The result is then divided one time by u_{ii}. Summarizing, for $i=n-1,n-2,\ldots,2,1$, we have

$n-i$	multiplications
$n-i$	subtractions
1	division

Assuming it takes 4 clock cycles for each addition, subtraction, and multiplication, and 16 clock cycles for division, then if C is the clock cycle time,

for $i = n$

computational time spent on addition=0

computational time spent on subtraction =0

computational time spent on multiplication =0

computational time spent on division $= 16C$

for $i = n-1, \ldots, 1$,

computational time spent on addition=0

computational time spent on subtraction

$$= (n-i)4C$$

computational time spent on multiplication

$$= (n-i)4C$$

computational time spent on division $= 16C$

The total computational time spent on back substitution then is

$$(0+0+0+16C) + \sum_{i=1}^{n-1}\left[0 + (n-i)4C + (n-i)4C + 16C\right]$$

$$= 16C + \sum_{i=1}^{n-1}(8n - 8i + 16)C$$

$$= 16C + \sum_{i=1}^{n-1}(8nC) - \sum_{i=1}^{n-1}(8iC) + \sum_{i=1}^{n-1}16C$$

$$= 16C + 8nC\sum_{i=1}^{n-1}1 - 8C\sum_{i=1}^{n-1}i + 16C\sum_{i=1}^{n-1}1$$

$$= 16C + 8nC(n-1) - 8C\frac{n-1}{2}(1+(n-1)) + 16C(n-1)$$

$$= C(4n^2 + 12n)$$

Questions

1. How much computational time does it take to conduct forward substitution?
2. How much computational time does it take to conduct forward elimination?
3. How much computational time does it take to find the inverse of a square matrix using LU decomposition?

References

LU Decomposition
http://numericalmethods.eng.usf.edu/topics/lu_decomposition.html