Solving First Order Linear ODE by Integrating Factor

Example

Solve the following ordinary differential equation using the integrating factor method.

$$\frac{dy}{dx} + 0.4y = 3e^{-x} \ , \ y(0) = 5 \qquad (1)$$

Solution

Writing the ordinary differential equation in the form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

then

$$P(x) = 0.4$$

$$Q(x) = 3e^{-x}$$

and the integrating factor is

$$u(x) = e^{\int P dx}$$

$$= e^{\int 0.4 dx}$$

$$= e^{0.4x}$$

Multiplying Eqn(1) by the integrating factor $u(x)$, we get

$$e^{0.4x}\left(\frac{dy}{dx} + 0.4y\right) = e^{0.4x}\left(3e^{-x}\right)$$

$$e^{0.4x}\frac{dy}{dx} + 0.4e^{0.4x}y = e^{0.4x}\left(3e^{-x}\right)$$

$$e^{0.4x}\frac{d}{dx}\left(y\right) + \frac{d}{dx}\left(e^{0.4x}\right)y = 3e^{-0.6x}$$

Using the formula that

$$u(x)\frac{dv(x)}{dx} + v(x)\frac{du(x)}{dx} = \frac{d}{dx}(u(x)v(x))$$

we get

$$\frac{d}{dx}\left(e^{0.4x}y\right) = 3e^{-0.6x}$$

$$e^{0.4x}y = \int 3e^{-0.6x}dx$$

$$= \frac{3e^{-0.6x}}{-0.6} + C \qquad (2)$$

Since

$$y(0) = 5$$

$$e^{0.4(0)}(5)=\frac{3e^{-0.6(0)}}{-0.6}+C$$
$$5=-5+C$$
$$C=10$$

Substituting the value of C in Eqn (2),

$$e^{0.4x}y=\frac{3e^{-0.6x}}{-0.6}+10$$
$$y=\frac{3e^{-0.6x}}{-0.6e^{0.4x}}+\frac{10}{e^{0.4x}}$$
$$=-5e^{-x}+10e^{-0.4x}$$

At $x=3$,

$$y(3)=-5e^{-3}+10e^{-0.4(3)}$$
$$=2.763$$