Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
LU Decomposition Method
Simultaneous Linear Equations
COMPLETE SOLUTION SET

1. The $[L][U]$ decomposition method is computationally more efficient than Naïve Gauss elimination for solving

(A) a single set of simultaneous linear equations.
(B) multiple sets of simultaneous linear equations with different coefficient matrices and the same right hand side vectors.
(C) multiple sets of simultaneous linear equations with the same coefficient matrix and different right hand side vectors.
(D) less than ten simultaneous linear equations.

Solution
The correct answer is (C).

An example can be demonstrated by finding the inverse of the matrix $[A]$. The problem of finding the inverse reduces to solving *n* sets of equations with the *n* columns of the identity matrix as the RHS vector. For calculations of each column of the inverse of the $[A]$ matrix, the coefficient matrix $[A]$ in the set of equations $[A][X]=[C]$ does not change. So if we use the LU Decomposition method, the $[A]=[L][U]$ decomposition needs to be done only once and the use of equations $[L][Z]=[C]$ and $[U][X]=[Z]$ still needs to be done *n* times.

So the total computational time required to find the inverse of a matrix using LU decomposition is approximately proportional to $\frac{n^3}{3}+n\times n^2 = \frac{4n^3}{3}$.

In comparison, if the Gaussian elimination method were applied to find the inverse of a matrix, the time would be approximately proportional to

$$n\left(\frac{n^3}{3}+\frac{n^2}{2}\right)=\frac{n^4}{3}+\frac{n^3}{2}$$

with the computational time of forward elimination approximately proportional to $\frac{n^3}{3}$, and back substitution approximately proportional to $\frac{n^2}{2}$.

For large values of n, $\frac{n^4}{3}+\frac{n^3}{2} \gg \frac{4n^3}{3}$.

2. The lower triangular matrix $[L]$ in the $[L][U]$ decomposition of the matrix given below

$$\begin{bmatrix} 25 & 5 & 4 \\ 10 & 8 & 16 \\ 8 & 12 & 22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

is

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0.40000 & 1 & 0 \\ 0.32000 & 1.7333 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 25 & 5 & 4 \\ 0 & 6 & 14.400 \\ 0 & 0 & -4.2400 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 10 & 1 & 0 \\ 8 & 12 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 & 0 \\ 0.40000 & 1 & 0 \\ 0.32000 & 1.5000 & 1 \end{bmatrix}$

Solution

The correct answer is (A).

We must first complete the first step of forward elimination.

$$\begin{bmatrix} 25 & 5 & 4 \\ 10 & 8 & 16 \\ 8 & 12 & 22 \end{bmatrix}$$

<u>First step</u>: Multiply *Row 1* by $\frac{10}{25} = 0.4$, and subtract the results from *Row 2*

$$[Row\,2] - [Row\,1] \times (0.4) = \begin{bmatrix} 25 & 5 & 4 \\ 0 & 6 & 14.4 \\ 8 & 12 & 22 \end{bmatrix}$$

Multiply *Row 1* by $\frac{8}{25} = 0.32$, and subtract the results from *Row 3*

$$[Row\,3]-[Row\,1]\times(0.32)=\begin{bmatrix} 25 & 5 & 4 \\ 0 & 6 & 14.4 \\ 0 & 10.4 & 20.72 \end{bmatrix}$$

To find ℓ_{21} and ℓ_{31}, what multiplier was used to make the a_{21} and a_{31} elements zero in the first step of forward elimination using the Naïve Gauss elimination method? They are

$$\ell_{21}=0.4$$

$$\ell_{31}=0.32$$

To find ℓ_{32}, what multiplier would be used to make the a_{32} element zero? Remember the a_{32} element is made zero in the second step of forward elimination.

So

$$\ell_{32}=\frac{10.4}{6}=1.7333$$

Hence

$$[L]=\begin{bmatrix} 1 & 0 & 0 \\ 0.40000 & 1 & 0 \\ 0.32000 & 1.7333 & 1 \end{bmatrix}$$

3. The upper triangular matrix $[U]$ in the $[L][U]$ decomposition of the matrix given below

$$\begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 12 & 22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

is

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0.40000 & 1 & 0 \\ 0.32000 & 1.7333 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 25 & 5 & 4 \\ 0 & 6 & 14.400 \\ 0 & 0 & -4.2400 \end{bmatrix}$

(C) $\begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 0 & -2 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0.2000 & 0.16000 \\ 0 & 1 & 2.4000 \\ 0 & 0 & -4.240 \end{bmatrix}$

Solution

The correct answer is (C).

The $[U]$ matrix is the same as the coefficient matrix that is found at the end of the forward elimination steps of the Naïve Gauss elimination method, that is

$$[U] = \begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 12 & 22 \end{bmatrix}$$

First step: The first step of forward elimination does not need to be conducted as a_{21} and a_{31} are already zero.

Second step: Multiply *Row 2* by $\frac{12}{8} = 1.5$ and subtract the results from *Row 3*

$$[Row\,3]-[Row\,2]\times(1.5)=\begin{bmatrix}25 & 5 & 4\\ 0 & 8 & 16\\ 0 & 0 & -2\end{bmatrix}$$

Thus,

$$[U]=\begin{bmatrix}25 & 5 & 4\\ 0 & 8 & 16\\ 0 & 0 & -2\end{bmatrix}$$

4. For a given 2000×2000 matrix $[A]$, assume that it takes about 15 seconds to find the inverse of $[A]$ by the use of the $[L][U]$ decomposition method, that is, finding the $[L][U]$ once, and then doing forward substitution and back substitution 2000 times using the 2000 columns of the identity matrix as the right hand side vector. The approximate time, in seconds, that it will take to find the inverse if found by repeated use of the Naive Gauss elimination method, that is, doing forward elimination and back substitution 2000 times by using the 2000 columns of the identity matrix as the right hand side vector is most nearly

(A) 300
(B) 1500
(C) 7500
(D) 30000

Solution
The correct solution is (C).

The computational times for finding the inverse of $[A]$ by Naïve Gaussian elimination and LU decomposition are proportional to $\frac{n^4}{3}+\frac{n^3}{2}$ and $\frac{4n^3}{3}$, respectively. Hence the ratio of computational time is

$$R \approx \frac{\frac{n^4}{3}+\frac{n^3}{2}}{\frac{4n^3}{3}}$$

For a 2000×2000 matrix

$$R \approx \frac{\frac{2000^4}{3}+\frac{2000^3}{2}}{4 \times \frac{2000^3}{3}}$$

$$= \frac{\frac{1.600 \times 10^{13}}{3}+\frac{8.000 \times 10^{9}}{2}}{\frac{3.200 \times 10^{10}}{3}}$$

$$= \frac{5.3373 \times 10^{12}}{1.0667 \times 10^{10}}$$

$$= 500.38$$

Since it takes 15 seconds to find the inverse by LU decomposition, it would take
$500.38 \times 15 = 7505.6$ seconds
to find the inverse by using Naïve Gaussian elimination.

5. The algorithm for solving a set of n equations $[A][X]=[C]$, where $[A]=[L][U]$ involves solving $[L][Z]=[C]$ by forward substitution. The algorithm to solve $[L][Z]=[C]$ is given by

(A)
```
$z_1 = c_1 / l_{11}$
for $i$ from 2 to $n$ do
sum = 0
for $j$ from 1 to $i$ do
sum = sum + $l_{ij} * z_j$
end do
$z_i = (c_i - sum)/ l_{ii}$
end do
```

(B)
```
$z_1 = c_1 / l_{11}$
for $i$ from 2 to $n$ do
sum = 0
for $j$ from 1 to $(i-1)$ do
sum = sum + $l_{ij} * z_j$
end do
$z_i = (c_i - sum)/ l_{ii}$
end do
```

(C)
```
$z_1 = c_1 / l_{11}$
for $i$ from 2 to $n$ do
for $j$ from 1 to $(i-1)$ do
sum = sum + $l_{ij} * z_j$
end do
$z_i = (c_i - sum)/ l_{ii}$
end do
```

(D)
```
for $i$ from 2 to $n$ do
sum = 0
for $j$ from 1 to $(i-1)$ do
sum = sum + $l_{ij} * z_j$
end do
$z_i = (c_i - sum)/ l_{ii}$
end do
```

Solution

The correct answer is (B).

Since $[L][Z]=[C]$ is written as

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ \ell_{21} & 1 & & \vdots \\ \ell_{31} & & \ddots & 0 \\ \vdots & & & \\ \ell_{n1} & \ell_{n2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

We know that the first step of forward substitution is

$$z_1 = c_1 / l_{11}$$

The next equations would be

$$z_2 = \frac{c_2 - \ell_{21} \times z_1}{\ell_{22}}$$

$$z_3 = \frac{c_3 - (\ell_{31} \times z_1 + \ell_{32} \times z_2)}{\ell_{33}}$$

$$\vdots$$

$$z_i = \frac{c_i - \sum_{j=1}^{i-1} (\ell_{ij} \times z_j)}{\ell_{ii}}$$

Thus, the algorithm is

```
$z_1 = c_1 / l_{11}$
for i from 2 to n do
  sum = 0                      (the variable *sum* must be cleared each time through the loop)
  for j from 1 to (i-1) do     (this loop finds $\sum_{j=1}^{i-1} (\ell_{ij} \times z_j)$)
     sum = sum + $l_{ij} * z_j$
  end do
  $z_i = (c_i - \text{sum}) / l_{ii}$
end do
```

6. To solve boundary value problems, the finite difference method is used resulting in simultaneous linear equations with tridiagonal coefficient matrices. These are solved using the specialized $[L][U]$ decomposition method. The set of equations in matrix form with a tridiagonal coefficient matrix for

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2,\ y(0) = 0,\ y(12) = 0,$$

using the finite difference method with a second order accurate central divided difference method and a step size of $h = 4$ is

(A) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.0625 & 0.125 & 0.0625 & 0 \\ 0 & 0.0625 & 0.125 & 0.0625 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 16.0 \\ 16.0 \\ 0 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.0625 & -0.125 & 0.0625 & 0 \\ 0 & 0.0625 & -0.125 & 0.0625 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 16.0 \\ 16.0 \\ 0 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0.0625 & -0.125 & 0.0625 & 0 \\ 0 & 0.0625 & -0.125 & 0.0625 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 16.0 \\ 16.0 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0.0625 & 0.125 & 0.0625 & 0 \\ 0 & 0.0625 & 0.125 & 0.0625 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 16.0 \\ 16.0 \end{bmatrix}$

Solution

The correct answer is (B).

The first equation and is

$$y(0) = 0$$

or

$$y_1 = 0$$

The approximation for $\frac{d^2y}{dx^2}$ with a second order accurate central divided difference method is

$$\frac{d^2y}{dx^2} \approx \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

Hence

$$\frac{y_{i-1}-2y_i+y_{i+1}}{h^2} \approx 6x_i - 0.5x_i^2$$

Where $x_{i+1} = x_i + h$

At node $i = 2,\ x_2 = 0 + 4 = 4$

$$\frac{y_1 - 2y_2 + y_3}{4^2} = 6 \times 4 - 0.5 \times 4^2$$

$$0.0625y_1 - 0.125y_2 + 0.0625y_3 = 16$$

At node $i = 3,\ x_2 = 4 + 4 = 8$

$$\frac{y_2 - 2y_3 + y_4}{4^2} = 6 \times 8 - 0.5 \times 8^2$$

$$0.0625y_2 - 0.125y_3 + 0.0625y_4 = 16$$

The final equation is

$$y(12) = 0$$

or

$$y_4 = 0$$

Written in matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.0625 & -0.125 & 0.0625 & 0 \\ 0 & 0.0625 & -0.125 & 0.0625 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 16.0 \\ 16.0 \\ 0 \end{bmatrix}$$