

## Chapter 05.03

# Newton's Divided Difference Interpolation

After reading this chapter, you should be able to:

1. derive Newton's divided difference method of interpolation,
2. apply Newton's divided difference method of interpolation, and
3. apply Newton's divided difference method interpolants to find derivatives and integrals.

### What is interpolation?

Many times, data is given only at discrete points such as  $(x_0, y_0)$ ,  $(x_1, y_1)$ , ...,  $(x_{n-1}, y_{n-1})$ ,  $(x_n, y_n)$ . So, how then does one find the value of  $y$  at any other value of  $x$ ? Well, a continuous function  $f(x)$  may be used to represent the  $n+1$  data values with  $f(x)$  passing through the  $n+1$  points (Figure 1). Then one can find the value of  $y$  at any other value of  $x$ . This is called *interpolation*.

Of course, if  $x$  falls outside the range of  $x$  for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function  $f(x)$  should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

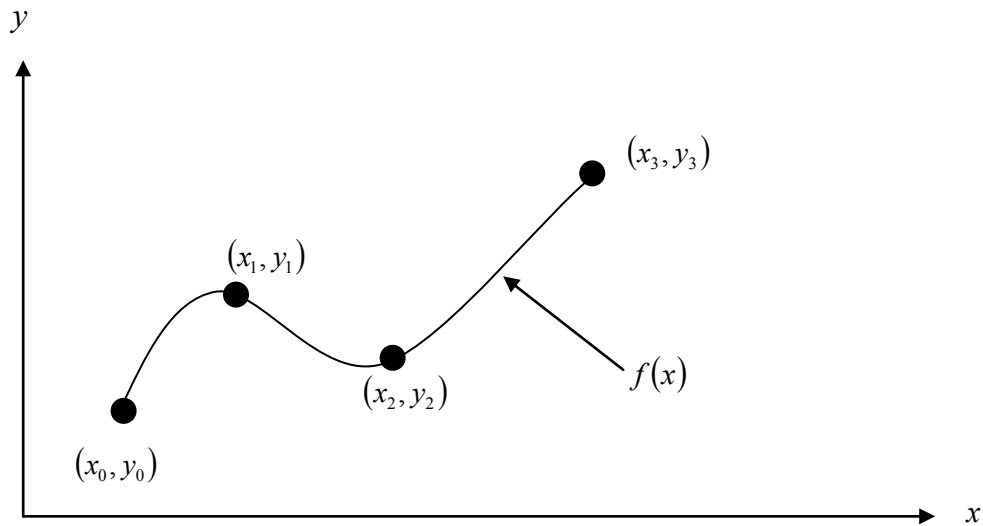
- (A) evaluate,
- (B) differentiate, and
- (C) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order  $n$  that passes through the  $n+1$  points. One of the methods of interpolation is called Newton's divided difference polynomial method. Other methods include the direct method and the Lagrangian interpolation method. We will discuss Newton's divided difference polynomial method in this chapter.

### Newton's Divided Difference Polynomial Method

To illustrate this method, linear and quadratic interpolation is presented first. Then, the general form of Newton's divided difference polynomial method is presented. To illustrate the general form, cubic interpolation is shown in Figure 1.



**Figure 1** Interpolation of discrete data.

### Linear Interpolation

Given  $(x_0, y_0)$  and  $(x_1, y_1)$ , fit a linear interpolant through the data. Noting  $y = f(x)$  and  $y_1 = f(x_1)$ , assume the linear interpolant  $f_1(x)$  is given by (Figure 2)

$$f_1(x) = b_0 + b_1(x - x_0)$$

Since at  $x = x_0$ ,

$$f_1(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) = b_0$$

and at  $x = x_1$ ,

$$\begin{aligned} f_1(x_1) &= f(x_1) = b_0 + b_1(x_1 - x_0) \\ &= f(x_0) + b_1(x_1 - x_0) \end{aligned}$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So

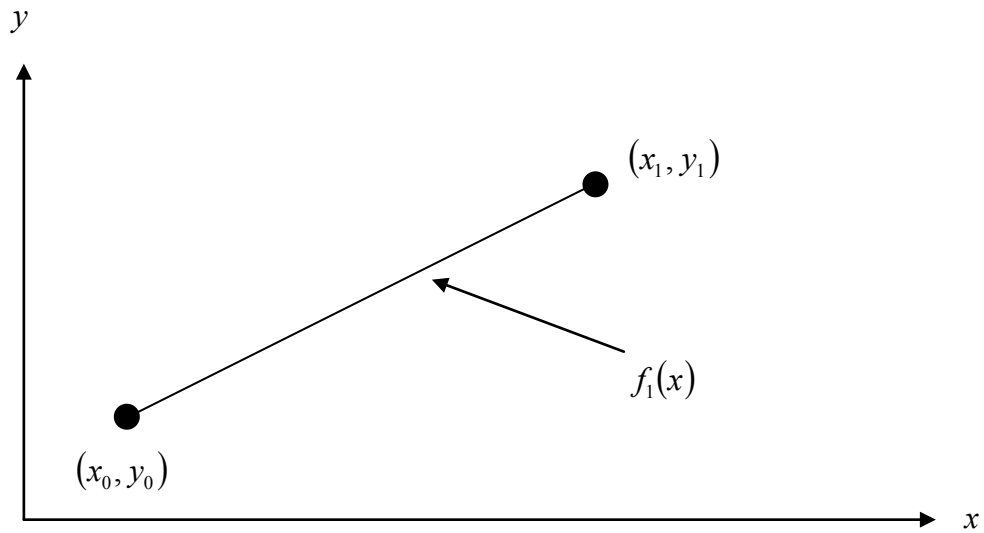
$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

giving the linear interpolant as

$$f_1(x) = b_0 + b_1(x - x_0)$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$



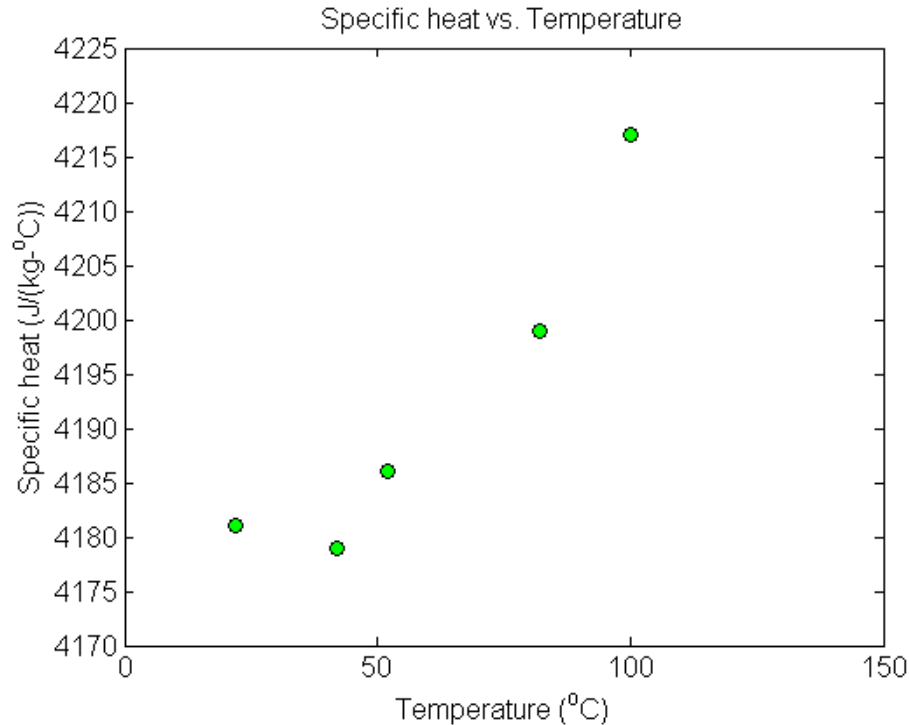
**Figure 2** Linear interpolation.

### Example 1

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at  $61^\circ\text{C}$ . The specific heat of water is given as a function of time in Table 1.

**Table 1** Specific heat of water as a function of temperature.

Temperature, $T$ ( $^\circ\text{C}$ )	Specific heat, $C_p$ $\left(\frac{\text{J}}{\text{kg} - ^\circ\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217



**Figure 3** Specific heat of water vs. temperature.

Determine the value of the specific heat at  $T = 61^\circ\text{C}$  using Newton's divided difference method of interpolation and a first order polynomial.

### Solution

For linear interpolation, the specific heat is given by

$$C_p(T) = b_0 + b_1(T - T_0)$$

Since we want to find the velocity at  $T = 61^\circ\text{C}$ , and we are using a first order polynomial we need to choose the two data points that are closest to  $T = 61^\circ\text{C}$  that also bracket  $T = 61^\circ\text{C}$  to evaluate it. The two points are  $T = 52$  and  $T = 82$ .

Then

$$T_0 = 52, \quad C_p(T_0) = 4186$$

$$T_1 = 82, \quad C_p(T_1) = 4199$$

gives

$$b_0 = C_p(T_0)$$

$$= 4186$$

$$b_1 = \frac{C_p(T_1) - C_p(T_0)}{T_1 - T_0}$$

$$= \frac{4199 - 4186}{82 - 52}$$

$$= 0.43333$$

Hence

$$\begin{aligned} C_p(T) &= b_0 + b_1(T - T_0) \\ &= 4186 + 0.43333(T - 52), \quad 52 \leq T \leq 82 \end{aligned}$$

At  $T = 61$ ,

$$\begin{aligned} C_p(61) &= 4186 + 0.43333(61 - 52) \\ &= 4189.9 \frac{\text{J}}{\text{kg} - ^\circ\text{C}} \end{aligned}$$

If we expand

$$C_p(T) = 4186 + 0.43333(T - 52), \quad 52 \leq T \leq 82$$

we get

$$C_p(T) = 4163.5 + 0.43333T, \quad 52 \leq T \leq 82$$

and this is the same expression as obtained in the direct method.

### Quadratic Interpolation

Given  $(x_0, y_0)$ ,  $(x_1, y_1)$ , and  $(x_2, y_2)$ , fit a quadratic interpolant through the data. Noting  $y = f(x)$ ,  $y_0 = f(x_0)$ ,  $y_1 = f(x_1)$ , and  $y_2 = f(x_2)$ , assume the quadratic interpolant  $f_2(x)$  is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

At  $x = x_0$ ,

$$\begin{aligned} f_2(x_0) &= f(x_0) = b_0 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1) \\ &= b_0 \end{aligned}$$

$$b_0 = f(x_0)$$

At  $x = x_1$

$$\begin{aligned} f_2(x_1) &= f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1) \\ f(x_1) &= f(x_0) + b_1(x_1 - x_0) \end{aligned}$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

At  $x = x_2$

$$\begin{aligned} f_2(x_2) &= f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \\ f(x_2) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \end{aligned}$$

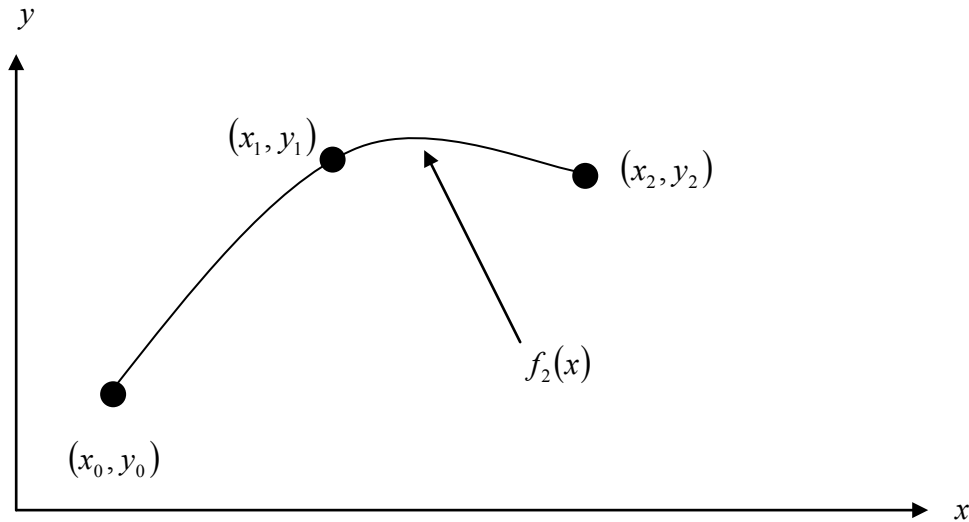
Giving

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Hence the quadratic interpolant is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}(x - x_0)(x - x_1)$$



**Figure 4** Quadratic interpolation.

### Example 2

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at  $61^\circ\text{C}$ . The specific heat of water is given as a function of time in Table 2.

**Table 2** Specific heat of water as a function of temperature.

Temperature, $T$ ( $^\circ\text{C}$ )	Specific heat, $C_p$ $\left(\frac{\text{J}}{\text{kg} - ^\circ\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

Determine the value of the specific heat at  $T = 61^\circ\text{C}$  using Newton's divided difference method of interpolation and a second order polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

### Solution

For quadratic interpolation, the specific heat is given by

$$C_p(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1)$$

Since we want to find the specific heat at  $T = 61^\circ\text{C}$ , and we are using a second order polynomial, we need to choose the three data points that are closest to  $T = 61^\circ\text{C}$  that also bracket  $T = 61^\circ\text{C}$  to evaluate it. The three points are  $T_0 = 42$ ,  $T_1 = 52$ , and  $T_2 = 82$ .

Then

$$T_0 = 42, \quad C_p(T_0) = 4179$$

$$T_1 = 52, \quad C_p(T_1) = 4186$$

$$T_2 = 82, \quad C_p(T_2) = 4199$$

gives

$$b_0 = C_p(T_0)$$

$$= 4179$$

$$b_1 = \frac{C_p(T_1) - C_p(T_0)}{T_1 - T_0}$$

$$= \frac{4186 - 4179}{52 - 42}$$

$$= 0.7$$

$$b_2 = \frac{\frac{C_p(T_2) - C_p(T_1)}{T_2 - T_1} - \frac{C_p(T_1) - C_p(T_0)}{T_1 - T_0}}{T_2 - T_0}$$

$$= \frac{\frac{4199 - 4186}{82 - 52} - \frac{4186 - 4179}{52 - 42}}{82 - 42}$$

$$= \frac{0.43333 - 0.7}{40}$$

$$= -6.6667 \times 10^{-3}$$

Hence

$$C_p(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1)$$

$$= 4179 + 0.7(T - 42) - 6.6667 \times 10^{-3}(T - 42)(T - 52), \quad 42 \leq T \leq 82$$

At  $T = 61$ ,

$$C_p(61) = 4179 + 0.7(61 - 42) - 6.6667 \times 10^{-3}(61 - 42)(61 - 52)$$

$$= 4191.2 \frac{\text{J}}{\text{kg} \cdot ^\circ\text{C}}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the first and second order polynomial is

$$|\epsilon_a| = \left| \frac{4191.2 - 4189.9}{4191.2} \right| \times 100$$

$$= 0.030063\%$$

If we expand

$$C_p(T) = 4179 + 0.7(T - 42) - 6.6667 \times 10^{-3}(T - 42)(T - 52), \quad 42 \leq T \leq 82$$

we get

$$C_p(T) = 4135.0 + 1.3267T - 6.6667 \times 10^{-3}T^2, \quad 42 \leq T \leq 82$$

This is the same expression obtained by the direct method.

### General Form of Newton's Divided Difference Polynomial

In the two previous cases, we found linear and quadratic interpolants for Newton's divided difference method. Let us revisit the quadratic polynomial interpolant formula

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Note that  $b_0$ ,  $b_1$ , and  $b_2$  are finite divided differences.  $b_0$ ,  $b_1$ , and  $b_2$  are the first, second, and third finite divided differences, respectively. We denote the first divided difference by

$$f[x_0] = f(x_0)$$

the second divided difference by

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and the third divided difference by

$$\begin{aligned} f[x_2, x_1, x_0] &= \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} \\ &= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} \end{aligned}$$

where  $f[x_0]$ ,  $f[x_1, x_0]$ , and  $f[x_2, x_1, x_0]$  are called bracketed functions of their variables enclosed in square brackets.

Rewriting,

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

This leads us to writing the general form of the Newton's divided difference polynomial for  $n + 1$  data points,  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ , as

$$f_n(x) = b_0 + b_1(x - x_0) + \dots + b_n(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$



$$\begin{aligned}
 b_2 &= f[x_2, x_1, x_0] \\
 &\vdots \\
 b_{n-1} &= f[x_{n-1}, x_{n-2}, \dots, x_0] \\
 b_n &= f[x_n, x_{n-1}, \dots, x_0]
 \end{aligned}$$

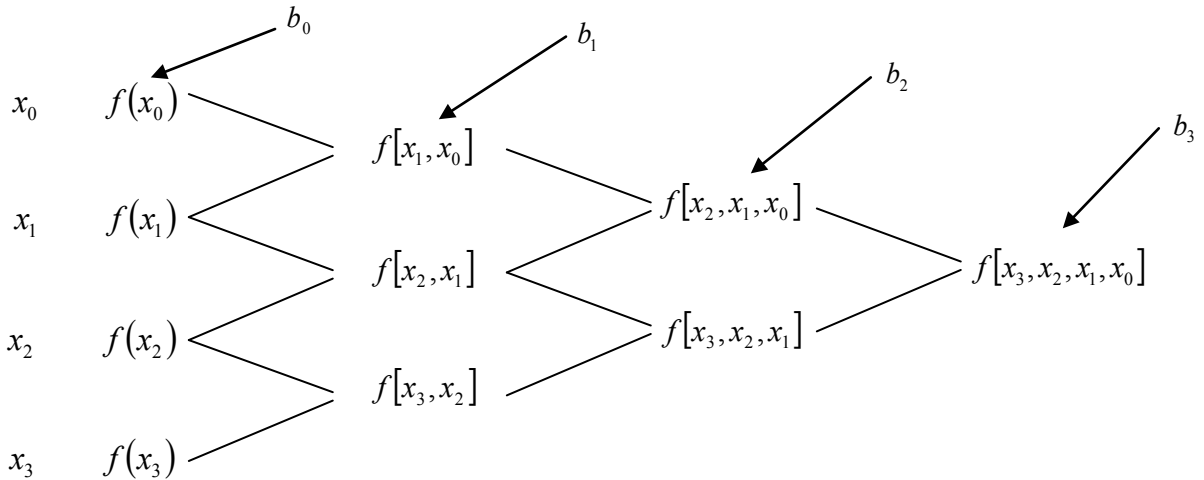
where the definition of the  $m^{\text{th}}$  divided difference is

$$\begin{aligned}
 b_m &= f[x_m, \dots, x_0] \\
 &= \frac{f[x_m, \dots, x_1] - f[x_{m-1}, \dots, x_0]}{x_m - x_0}
 \end{aligned}$$

From the above definition, it can be seen that the divided differences are calculated recursively.

For an example of a third order polynomial, given  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$ ,

$$\begin{aligned}
 f_3(x) &= f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) \\
 &\quad + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)
 \end{aligned}$$



**Figure 5** Table of divided differences for a cubic polynomial.

**Example 3**

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at 61°C. The specific heat of water is given as a function of time in Table 3.

**Table 3** Specific heat of water as a function of temperature.

Temperature, $T$ (°C)	Specific heat, $C_p$ $\left(\frac{\text{J}}{\text{kg} \cdot ^\circ\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

Determine the value of the specific heat at  $T = 61^\circ\text{C}$  using Newton's divided difference method of interpolation and a third order polynomial. Find the absolute relative approximate error for the third order polynomial approximation.

**Solution**

For a third order polynomial, the specific heat profile is given by

$$C_p(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1) + b_3(T - T_0)(T - T_1)(T - T_2)$$

Since we want to find the specific heat at  $T = 61^\circ\text{C}$ , and we are using a third order polynomial, we need to choose the four data points that are closest to  $T = 61^\circ\text{C}$  that also bracket  $T = 61^\circ\text{C}$ . The four data points are  $T_0 = 42$ ,  $T_1 = 52$ ,  $T_2 = 82$  and  $T_3 = 100$ .

(Choosing the four points as  $T_0 = 22$ ,  $T_1 = 42$ ,  $T_2 = 52$  and  $T_3 = 82$  is equally valid.)

$$T_0 = 42, \quad C_p(T_0) = 4179$$

$$T_1 = 52, \quad C_p(T_1) = 4186$$

$$T_2 = 82, \quad C_p(T_2) = 4199$$

$$T_3 = 100, \quad C_p(T_3) = 4217$$

then

$$b_0 = C_p[T_0]$$

$$= C_p(T_0)$$

$$= 4179$$

$$b_1 = C_p[T_1, T_0]$$

$$= \frac{C_p(T_1) - C_p(T_0)}{T_1 - T_0}$$

$$= \frac{4186 - 4179}{52 - 42}$$

$$= 0.7$$

$$b_2 = C_p[T_2, T_1, T_0]$$

$$= \frac{C_p[T_2, T_1] - C_p[T_1, T_0]}{T_2 - T_0}$$

$$\begin{aligned} C_p[T_2, T_1] &= \frac{C_p(T_2) - C_p(T_1)}{T_2 - T_1} \\ &= \frac{4199 - 4186}{82 - 52} \\ &= 0.43333 \end{aligned}$$

$$C_p[T_1, T_0] = 0.7$$

$$\begin{aligned} b_2 &= \frac{C_p[T_2, T_1] - C_p[T_1, T_0]}{T_2 - T_0} \\ &= \frac{0.43333 - 0.7}{82 - 42} \\ &= -6.6667 \times 10^{-3} \end{aligned}$$

$$\begin{aligned} b_3 &= C_p[T_3, T_2, T_1, T_0] \\ &= \frac{C_p[T_3, T_2, T_1] - C_p[T_2, T_1, T_0]}{T_3 - T_0} \end{aligned}$$

$$C_p[T_3, T_2, T_1] = \frac{C_p[T_3, T_2] - C_p[T_2, T_1]}{T_3 - T_1}$$

$$\begin{aligned} C_p[T_3, T_2] &= \frac{C_p(T_3) - C_p(T_2)}{T_3 - T_2} \\ &= \frac{4217 - 4199}{100 - 82} \\ &= 1 \end{aligned}$$

$$\begin{aligned} C_p[T_2, T_1] &= \frac{C_p(T_2) - C_p(T_1)}{T_2 - T_1} \\ &= \frac{4199 - 4186}{82 - 52} \\ &= 0.43333 \end{aligned}$$

$$\begin{aligned} C_p[T_3, T_2, T_1] &= \frac{C_p[T_3, T_2] - C_p[T_2, T_1]}{T_3 - T_1} \\ &= \frac{1 - 0.43333}{100 - 52} \\ &= 0.011806 \end{aligned}$$

$$C_p[T_2, T_1, T_0] = -6.6667 \times 10^{-3}$$

$$\begin{aligned} b_3 &= C_p[T_3, T_2, T_1, T_0] \\ &= \frac{C_p[T_3, T_2, T_1] - C_p[T_2, T_1, T_0]}{T_3 - T_0} \end{aligned}$$

$$= \frac{0.011806 + 6.6667 \times 10^{-3}}{100 - 42}$$

$$= 3.1849 \times 10^{-4}$$

Hence

$$C_p(T) = b_0 + b_1(T - T_0) + b_2(T - T_0)(T - T_1) + b_3(T - T_0)(T - T_1)(T - T_2)$$

$$= 4179 + 0.7(T - 42) - 6.6667 \times 10^{-3}(T - 42)(T - 52)$$

$$+ 3.1849 \times 10^{-4}(T - 42)(T - 52)(T - 82), \quad 42 \leq T \leq 100$$

At  $T = 61$ ,

$$C_p(61) = 4179 + 0.7(61 - 42) - 6.6667 \times 10^{-3}(61 - 42)(61 - 52)$$

$$+ 3.1849 \times 10^{-4}(61 - 42)(61 - 52)(61 - 82)$$

$$= 4190.0 \frac{\text{J}}{\text{kg} - ^\circ\text{C}}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the second and third order polynomial is

$$|\epsilon_a| = \left| \frac{4190.0 - 4191.2}{4190.0} \right| \times 100$$

$$= 0.027295\%$$

If we expand

$$C_p(T) = 4179 + 0.7(T - 42) - 6.6667 \times 10^{-3}(T - 42)(T - 52)$$

$$+ 3.1849 \times 10^{-4}(T - 42)(T - 52)(T - 82), \quad 42 \leq T \leq 100$$

we get

$$C_p(T) = 4078.0 + 4.4771T - 0.06272T^2 + 3.1849 \times 10^{-4}T^3, \quad 42 \leq T \leq 100$$

This is the same expression as obtained in the direct method.

INTERPOLATION	
Topic	Newton's Divided Difference Interpolation
Summary	Textbook notes on Newton's divided difference interpolation.
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