Chapter 02.02

Physical Problem for Differentiation

A rectangular plate under a uniform stress, σ_0, results in the stress state of

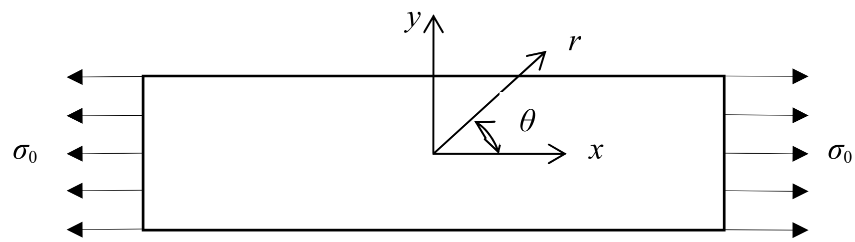

Figure 1 Rectangular plate under uniform stress.

Where

$$\sigma_{xx} = \sigma_0$$
$$\sigma_{yy} = 0$$
$$\sigma_{zz} = 0$$

If the plate now has changed in the cross-section, the normal stresses will increase as a result of these changes. For example, if there is a circular hole in the plate (Figure 2) and it is subjected to the same uniform stress, σ_0, then stresses in the plate would increase. For example, if the radius of the hole in the plate was small compared to the length and width of the plate, the solution can be found analytically for $\sigma_{xx} = 3\sigma_0$ at $r = a, \theta = \pm\frac{\pi}{2}$.

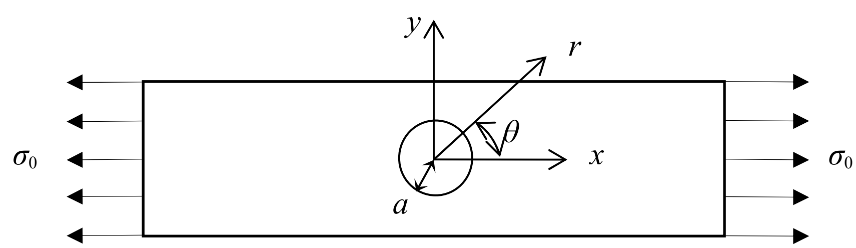

Figure 2 Rectangular plate with circular hole of radius *a* under uniform stress.

This in fact is the highest normal stress that occurs in such a case. To specify such stresses we define a term called stress concentration, *k*, that is

$$k = \frac{\text{Maximum Stress}}{\text{Nominal Stress}}$$

So in this case

$$k = \frac{3\sigma_0}{\sigma_0}$$
$$= 3$$

Now what happens when the radius of the hole is not small as compared to the length and width of the plate? In most cases, the stresses cannot be found analytically, and we need to solve the problem using numerical techniques. One such method is called the finite difference method. Here is how it works for this problem. The partial differential equations for the plane stress problem are given by

$$\frac{\partial^2 u}{\partial x^2}+\frac{(1-\upsilon)}{2}\frac{\partial^2 v}{\partial y^2}+\frac{(1+\upsilon)}{2}\frac{\partial^2 v}{\partial x \partial y}=0 \qquad (1)$$

$$\frac{\partial^2 v}{\partial y^2}+\frac{(1-\upsilon)}{2}\frac{\partial^2 u}{\partial x^2}+\frac{(1+\upsilon)}{2}\frac{\partial^2 u}{\partial x \partial y}=0 \qquad (2)$$

Where

$u =$ Displacement in x - direction

$v =$ Displacement in y - direction

$\upsilon =$ Poisson's ratio

To apple the finite difference method, the rectangular plate with the hole would be divided into (m-1) segments along the x-axis and (n-1) segments along the y-axis as shown in Figure 3. This results in nodes in the plate.

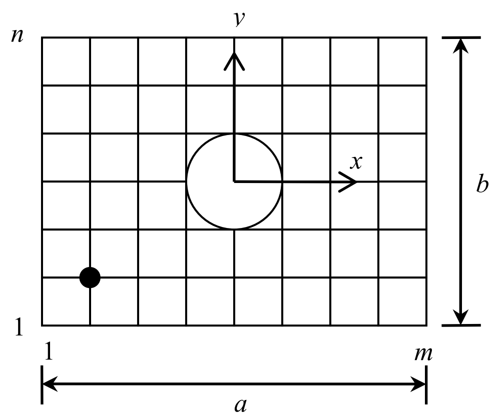

Figure 3 Finite Difference Nodes of Rectangular Plate with Hole.

Now at each node such as a general node (*i*,*j*) where the surrounding nodes do not have a node on the hole, the forward divided difference approximations can be applied as

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} \approx \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{(\Delta x)^2} \qquad (3)$$

$$\left.\frac{\partial^2 v}{\partial y^2}\right|_{i,j} \approx \frac{v_{i,j+1}-2v_{i,j}+v_{i,j-1}}{(\Delta y)^2} \qquad (4)$$

$$\left.\frac{\partial^2 v}{\partial x \partial y}\right|_{i,j} \approx \frac{v_{i+1,j+1}-v_{i+1,j-1}-v_{i-1,j+1}-v_{i-1,j-1}}{(\Delta x)(\Delta y)} \qquad (5)$$

$$\left.\frac{\partial^2 u}{\partial x \partial y}\right|_{i,j} \approx \frac{u_{i+1,j+1}-u_{i+1,j-1}-u_{i-1,j+1}-u_{i-1,j-1}}{(\Delta x)(\Delta y)} \qquad (6)$$

Substituting these approximations (equations 3-6) in equation (1) and (2) gives

$$\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{(\Delta x)^2}+\frac{(1-\upsilon)}{2}\left[\frac{v_{i,j+1}-2v_{i,j}+v_{i,j-1}}{(\Delta y)^2}\right]+\frac{(1+\upsilon)}{2}\left[\frac{v_{i+1,j+1}-v_{i+1,j-1}-v_{i-1,j+1}-v_{i-1,j-1}}{(\Delta x)(\Delta y)}\right]=0$$

$$\frac{v_{i,j+1}-2v_{i,j}+v_{i,j-1}}{(\Delta y)^2}+\frac{(1-\upsilon)}{2}\left[\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{(\Delta x)^2}\right]+\frac{(1+\upsilon)}{2}\left[\frac{u_{i+1,j+1}-u_{i+1,j-1}-u_{i-1,j+1}-u_{i-1,j-1}}{(\Delta x)(\Delta y)}\right]=0$$

Similarly one would write equations for the nodes that are around the hole and also for ones that are surrounded by the nodes on the hole. This is out of the scope of this topic but needs to be appreciated. We will also skip the application of boundary conditions at the outer four edges of the plate as well as the radius of the hole. However, one needs to realize that once all the equations are defined, we get as many unknowns (6: the displacement in the x- and y-directions, at each node) as we have equations by the approximations to equations (1) and (2), as well as the boundary conditions. This results in solving simultaneous linear equations. After the displacements are calculated, one can find stress and strain in the plate to consequently find the stress concentration. As an example, the problem description of one such case is given below.

To find the stresses and hence the stress concentration factor, one needs to find the derivatives of these displacements. In Table 1 the radial displacements u are given along the y-axis. The radius of the hole is 1.0 cm.

a) At $x=0$, if the radial strain ε_r is given by $\varepsilon_r=\frac{\partial u}{\partial r}$, find the radial strain at $r=1.1\,\text{cm}$ using the forward divided difference method.
b) If the tangential strain at $r=1.1\,\text{cm}$, $\theta=90°$ is given to you as $\varepsilon_\theta=0.0029733$, find the hoop stress, σ_θ, at $r=1.1\,\text{cm}$, $\theta=90°$ if $\sigma_\theta=\frac{E}{1-\nu^2}(\varepsilon_r+\nu\varepsilon_\theta)$, where $E=200\text{ GPa}$ and $\nu=0.3$.

Table 2 Radial displacement as a function of location.

r (cm)	u (cm)
1.0	–0.0010000
1.1	–0.0010689
1.2	–0.0011088
1.3	–0.0011326
1.4	–0.0011474
1.5	–0.0011574
1.6	–0.0011650
1.7	–0.0011718
1.8	–0.0011785
1.9	–0.0011857

Differentiation	
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Summary	Stress Concentration Around a Hole
Major	Civil Engineering
Authors	Autar Kaw
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Web Site	http://numericalmethods.eng.usf.edu