Chapter 03.03
Bisection Method of Solving a Nonlinear Equation

After reading this chapter, you should be able to:

1. *follow the algorithm of the bisection method of solving a nonlinear equation,*
2. *use the bisection method to solve examples of finding roots of a nonlinear equation, and*
3. *enumerate the advantages and disadvantages of the bisection method.*

What is the bisection method and what is it based on?

One of the first numerical methods developed to find the root of a nonlinear equation $f(x) = 0$ was the bisection method (also called *binary-search* method). The method is based on the following theorem.

Theorem

An equation $f(x) = 0$, where $f(x)$ is a real continuous function, has at least one root between x_ℓ and x_u if $f(x_\ell)f(x_u) < 0$ (See Figure 1).

Note that if $f(x_\ell)f(x_u) > 0$, there may or may not be any root between x_ℓ and x_u (Figures 2 and 3). If $f(x_\ell)f(x_u) < 0$, then there may be more than one root between x_ℓ and x_u (Figure 4). So the theorem only guarantees one root between x_ℓ and x_u.

Bisection method

Since the method is based on finding the root between two points, the method falls under the category of bracketing methods.

Since the root is bracketed between two points, x_ℓ and x_u, one can find the mid-point, x_m between x_ℓ and x_u. This gives us two new intervals

1. x_ℓ and x_m, and
2. x_m and x_u.

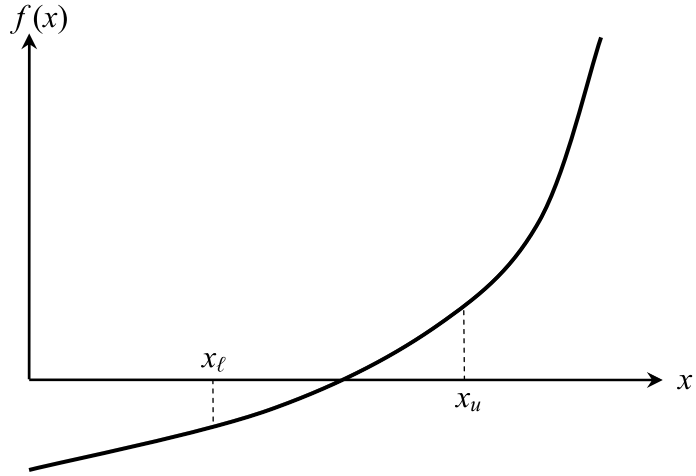

Figure 1 At least one root exists between the two points if the function is real, continuous, and changes sign.

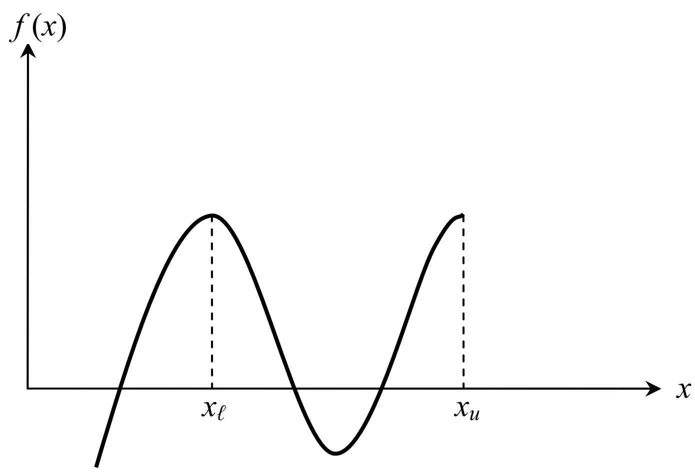

Figure 2 If the function $f(x)$ does not change sign between the two points, roots of the equation $f(x)=0$ may still exist between the two points.

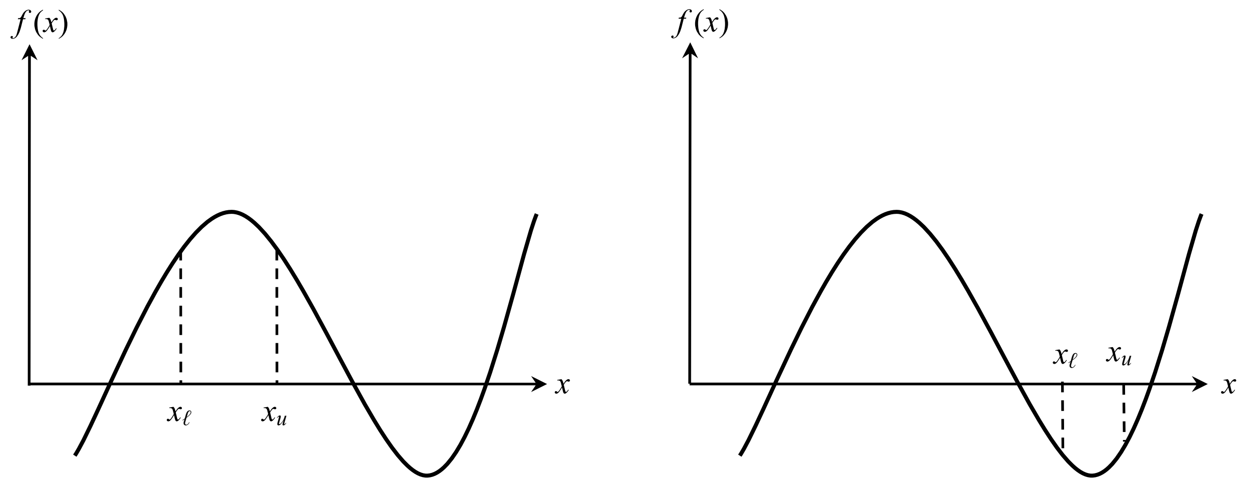

Figure 3 If the function $f(x)$ does not change sign between two points, there may not be any roots for the equation $f(x)=0$ between the two points.

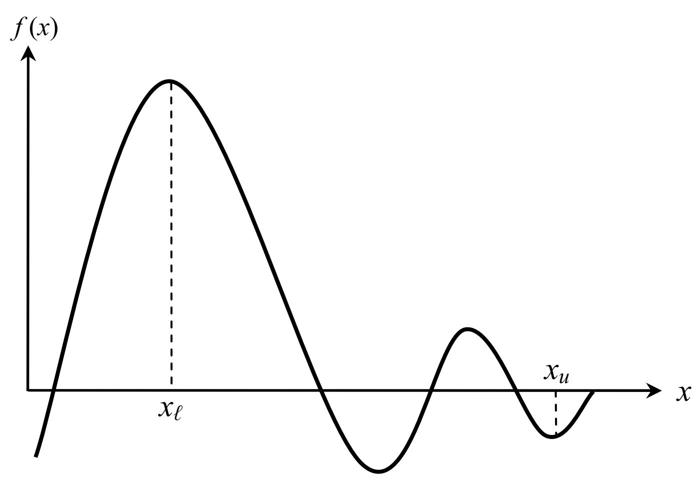

Figure 4 If the function $f(x)$ changes sign between the two points, more than one root for the equation $f(x)=0$ may exist between the two points.

Is the root now between x_ℓ and x_m or between x_m and x_u? Well, one can find the sign of $f(x_\ell)f(x_m)$, and if $f(x_\ell)f(x_m)<0$ then the new bracket is between x_ℓ and x_m, otherwise, it is between x_m and x_u. So, you can see that you are literally halving the interval. As one repeats this process, the width of the interval $[x_\ell, x_u]$ becomes smaller and smaller, and you can zero in to the root of the equation $f(x)=0$. The algorithm for the bisection method is given as follows.

Algorithm for the bisection method

The steps to apply the bisection method to find the root of the equation $f(x)=0$ are

1. Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)f(x_u)<0$, or in other words, $f(x)$ changes sign between x_ℓ and x_u.
2. Estimate the root, x_m, of the equation $f(x)=0$ as the mid-point between x_ℓ and x_u as
$$x_m = \frac{x_\ell + x_u}{2}$$
3. Now check the following
   a) If $f(x_\ell)f(x_m)<0$, then the root lies between x_ℓ and x_m; then $x_\ell = x_\ell$ and $x_u = x_m$.
   b) If $f(x_\ell)f(x_m)>0$, then the root lies between x_m and x_u; then $x_\ell = x_m$ and $x_u = x_u$.
   c) If $f(x_\ell)f(x_m)=0$; then the root is x_m. Stop the algorithm if this is true.
4. Find the new estimate of the root
$$x_m = \frac{x_\ell + x_u}{2}$$
Find the absolute relative approximate error as
$$\left|\in_a\right| = \left|\frac{x_m^{\text{new}} - x_m^{\text{old}}}{x_m^{\text{new}}}\right| \times 100$$
where
x_m^{new} = estimated root from present iteration
x_m^{old} = estimated root from previous iteration
5. Compare the absolute relative approximate error $\left|\in_a\right|$ with the pre-specified relative error tolerance $\in_s$. If $\left|\in_a\right| > \in_s$, then go to Step 3, else stop the algorithm. Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Example 1

Thermistors are temperature-measuring devices based on the principle that the thermistor material exhibits a change in electrical resistance with a change in temperature. By measuring the resistance of the thermistor material, one can then determine the temperature. For a 10K3A Betatherm thermistor,

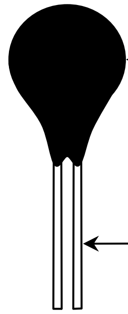

Figure 5 A typical thermistor.

the relationship between the resistance R of the thermistor and the temperature is given by

$$\frac{1}{T} = 1.129241\times10^{-3} + 2.341077\times10^{-4}\ln(R) + 8.775468\times10^{-8}\{\ln(R)\}^3$$

where T is in Kelvin and R is in ohms.

A thermistor error of no more than $\pm 0.01°\text{C}$ is acceptable. To find the range of the resistance that is within this acceptable limit at $19\,°\text{C}$, we need to solve

$$\frac{1}{19.01+273.15} = 1.129241\times10^{-3} + 2.341077\times10^{-4}\ln(R) + 8.775468\times10^{-8}\{\ln(R)\}^3$$

and

$$\frac{1}{18.99+273.15} = 1.129241\times10^{-3} + 2.341077\times10^{-4}\ln(R) + 8.775468\times10^{-8}\{\ln(R)\}^3$$

Use the bisection method of finding roots of equations to find the resistance R at $18.99°\text{C}$. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

Solution

Solving

$$\frac{1}{18.99+273.15} = 1.129241\times10^{-3} + 2.341077\times10^{-4}\ln(R) + 8.775468\times10^{-8}\{\ln(R)\}^3$$

we get

$$f(R) = 2.341077\times10^{-4}\ln(R) + 8.775468\times10^{-8}\{\ln(R)\}^3 - 2.293775\times10^{-3}$$

Lets us assume

$$R_\ell = 11000,\ R_u = 14000$$

Check if the function changes sign between R_ℓ and R_u.

$$f(R_\ell)=f(11000)$$
$$=2.341077\times10^{-4}\ln(11000)+8.775468\times10^{-8}\{\ln(11000)\}^3-2.293775\times10^{-3}$$
$$=-4.4536\times10^{-5}$$
$$f(R_u)=f(14000)$$
$$=2.341077\times10^{-4}\ln(14000)+8.775468\times10^{-8}\{\ln(14000)\}^3-2.293775\times10^{-3}$$
$$=1.7563\times10^{-5}$$

Hence

$$f(R_\ell)f(R_u)=f(11000)f(14000)=(-4.4536\times10^{-5})(1.7563\times10^{-5})<0$$

So there is at least one root between R_ℓ and R_u, that is, between 11000 and 14000.

Iteration 1

The estimate of the root is

$$R_m=\frac{R_\ell+R_u}{2}$$
$$=\frac{11000+14000}{2}$$
$$=12500$$

$$f(R_m)=f(12500)$$
$$=2.341077\times10^{-4}\ln(12500)+8.775468\times10^{-8}\{\ln(12500)\}^3-2.293775\times10^{-3}$$
$$=-1.1655\times10^{-5}$$
$$f(R_\ell)f(R_m)=f(11000)f(12500)=(-4.4536\times10^{-5})(-1.1655\times10^{-5})>0$$

Hence the root is bracketed between R_m and R_u, that is, between 12500 and 14000. So, the lower and upper limits of the new bracket are

$$R_\ell=12500,\ R_u=14000$$

At this point, the absolute relative approximate error $|\in_a|$ cannot be calculated as we do not have a previous approximation.

Iteration 2

The estimate of the root is

$$R_m=\frac{R_\ell+R_u}{2}$$
$$=\frac{12500+14000}{2}$$
$$=13250$$

$$f(R_m) = f(13250)$$
$$= 2.341077\times10^{-4}\ln(13250) + 8.775468\times10^{-8}\{\ln(13250)\}^3 - 2.293775\times10^{-3}$$
$$= 3.3599\times10^{-6}$$
$$f(R_\ell)f(R_m) = f(12500)f(13250) = (-1.1655\times10^{-5})(3.3599\times10^{-6}) < 0$$

Hence, the root is bracketed between R_ℓ and R_m, that is, between 12500 and 13250.
So the lower and upper limits of the new bracket are

$$R_\ell = 12500,\ R_u = 13250$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$|\in_a| = \left|\frac{R_m^{\text{new}} - R_m^{\text{old}}}{R_m^{\text{new}}}\right|\times100$$
$$= \left|\frac{13250-12500}{13250}\right|\times100$$
$$= 5.6604\%$$

None of the significant digits are at least correct in the estimated root

$$R_m = 13250$$

as the absolute relative approximate error is greater than 5%.

<u>Iteration 3</u>

$$R_m = \frac{R_\ell + R_u}{2}$$
$$= \frac{12500+13250}{2}$$
$$= 12875$$
$$f(R_m) = f(12875)$$
$$= 2.341077\times10^{-4}\ln(12875) + 8.775468\times10^{-8}\{\ln(12875)\}^3 - 2.293775\times10^{-3}$$
$$= -4.0403\times10^{-6}$$
$$f(R_\ell)f(R_m) = f(12500)f(12875) = ((-1.1654\times10^{-5}))(-4.0398\times10^{-6}) > 0$$

Hence, the root is bracketed between R_m and R_u, that is, between 12875 and 13250.
So, the lower and upper limits of the new bracket are

$$R_\ell = 12875,\ R_u = 13250$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 3 is

$$|\in_a| = \left|\frac{R_m^{\text{new}} - R_m^{\text{old}}}{R_m^{\text{new}}}\right|\times100$$
$$= \left|\frac{12875-13250}{12875}\right|\times100$$
$$= 2.9126\%$$

One of the significant digits is at least correct in the estimated root of the equation as the absolute relative approximate error is less than 5%.

Seven more iterations were conducted and these iterations are shown in the Table 1.

Table 1 Root of $f(x)=0$ as a function of the number of iterations for bisection method.

Iteration	R_ℓ	R_u	R_m	$\lvert\in_a\rvert\%$	$f(R_m)$
1	11000	14000	12500	----------	-1.1655×10^{-5}
2	12500	14000	13250	5.6604	3.3599×10^{-6}
3	12500	13250	12875	2.9126	-4.0403×10^{-6}
4	12875	13250	13063	1.4354	-3.1417×10^{-7}
5	13063	13250	13156	0.71259	1.5293×10^{-6}
6	13063	13156	13109	0.35757	6.0917×10^{-7}
7	13063	13109	13086	0.17910	1.4791×10^{-7}
8	13063	13086	13074	0.089633	-8.3022×10^{-8}
9	13074	13086	13080	0.044796	3.2470×10^{-8}
10	13074	13080	13077	0.022403	-2.5270×10^{-8}

At the end of the 10^{th} iteration,

$$\lvert\in_a\rvert = 0.022403\%$$

Hence the number of significant digits at least correct is given by the largest value of m for which

$$\lvert\in_a\rvert \le 0.5\times10^{2-m}$$

$$0.022403 \le 0.5\times10^{2-m}$$

$$0.044806 \le 10^{2-m}$$

$$\log(0.044806) \le 2-m$$

$$m \le 2-\log(0.044806) = 3.3487$$

So

$$m = 3$$

The number of significant digits at least correct in the estimated root 13077 is 3.

Advantages of bisection method

a) The bisection method is always convergent. Since the method brackets the root, the method is guaranteed to converge.
b) As iterations are conducted, the interval gets halved. So one can guarantee the error in the solution of the equation.

Drawbacks of bisection method

a) The convergence of the bisection method is slow as it is simply based on halving the interval.
b) If one of the initial guesses is closer to the root, it will take larger number of iterations to reach the root.

c) If a function $f(x)$ is such that it just touches the x-axis (Figure 6) such as

$$f(x) = x^2 = 0$$

it will be unable to find the lower guess, x_ℓ, and upper guess, x_u, such that

$$f(x_\ell)f(x_u) < 0$$

d) For functions $f(x)$ where there is a singularity[1] and it reverses sign at the singularity, the bisection method may converge on the singularity (Figure 7). An example includes

$$f(x) = \frac{1}{x}$$

where $x_\ell = -2$, $x_u = 3$ are valid initial guesses which satisfy

$$f(x_\ell)f(x_u) < 0$$

However, the function is not continuous and the theorem that a root exists is also not applicable.

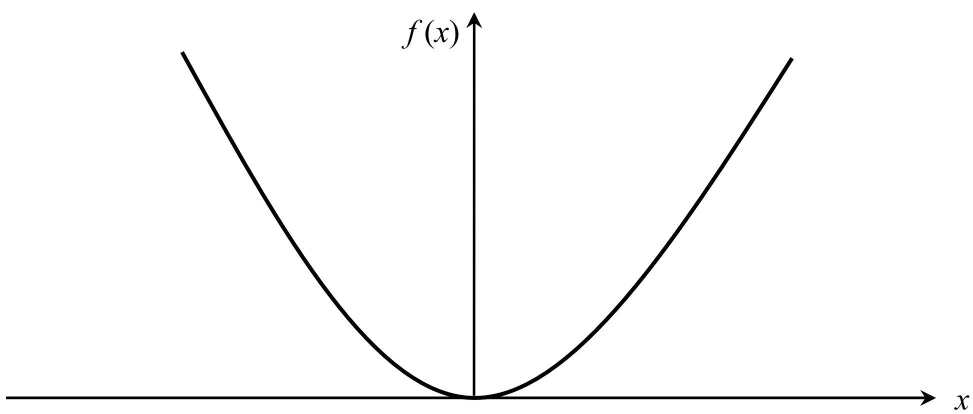

Figure 6 The equation $f(x) = x^2 = 0$ has a single root at $x = 0$ that cannot be bracketed.

[1] A singularity in a function is defined as a point where the function becomes infinite. For example, for a function

such as $1/x$, the point of singularity is $x=0$ as it becomes infinite.

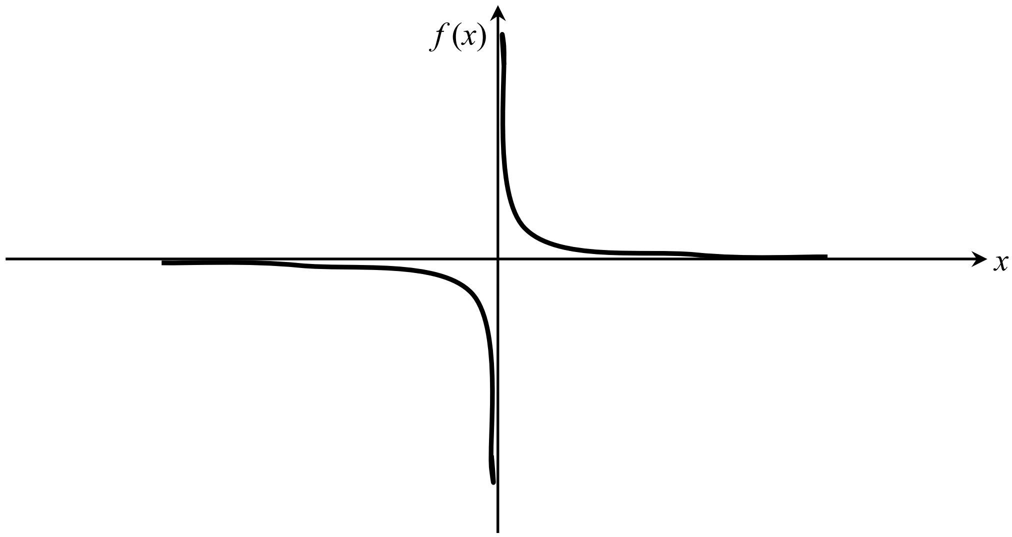

Figure 7 The equation $f(x)=\frac{1}{x}=0$ has no root but changes sign.

NONLINEAR EQUATIONS	
Topic	Bisection method of solving a nonlinear equation
Summary	These are textbook notes of bisection method of finding roots of nonlinear equation, including convergence and pitfalls.
Major	Electrical Engineering
Authors	Autar Kaw
Date	November 20, 2009
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