Chapter 07.00E

Physical Problem for Integration
Electrical Engineering

Today's consumer products rely heavily on small electronic circuits to provide much of their functionality. Each of these circuits is built around electrical components that, like all devices, vary in value due to variations in materials, manufacturing, and operating conditions. For example, capacitors are notoriously variable often operating as much as $+/-20\%$ from their nominal value. Clearly, a variation in capacitor value can have an affect on the operation of an electronic circuit. If the statistical properties of the capacitor and resistor value distributions are known, stochastic methods can be used to compute this effect. Unfortunately, this is often not the case and we are frequently left to use empirical methods to determine this effect. A common technique is to use a Monte Carlo method where a circuit will be built (or simulated) many times with different components from a lot and the resulting behavior analyzed. The goal will be to determine a confidence value that a randomly selected circuit built with these components will behave within a certain interval.

Consider the case of an op-amp based square wave generator as shown in Figure 1. Its operative behavior in terms of period is described by the equation (1, pp. 625-627]

$$T = 2RC\ln\left(1 + 2\frac{R_2}{R_1}\right)$$

Clearly, the period/frequency of this circuit is quite susceptible to variations in all the components[1] except the op-amp, resistor, R_3 and the zener diodes. How then can the likely behavior to all the possible combination of variations in the components R, C, R_1 and R_2 be determined? The answer is to apply a Monte Carlo method and some basic statistics.
If we assume that the statistical distribution of the circuit behavior is normal1 then a two-tailed confidence interval test can be used to determine the likelihood that the circuit will

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[1] This assumption should be tested, but that is beyond the scope of this example. See question 5 for more information.

behave acceptably [2, pp. 113-27].   An examination of the literature will show that a confidence interval can be established through the use of the z-test. For a sample of size n with known standard deviation σ it can be determined with confidence $(1-\alpha)$ that the measured value will stay in the following range

$$\bar{x} - z(\alpha/2)\sigma < value < \bar{x} + z(\alpha/2)\sigma,$$

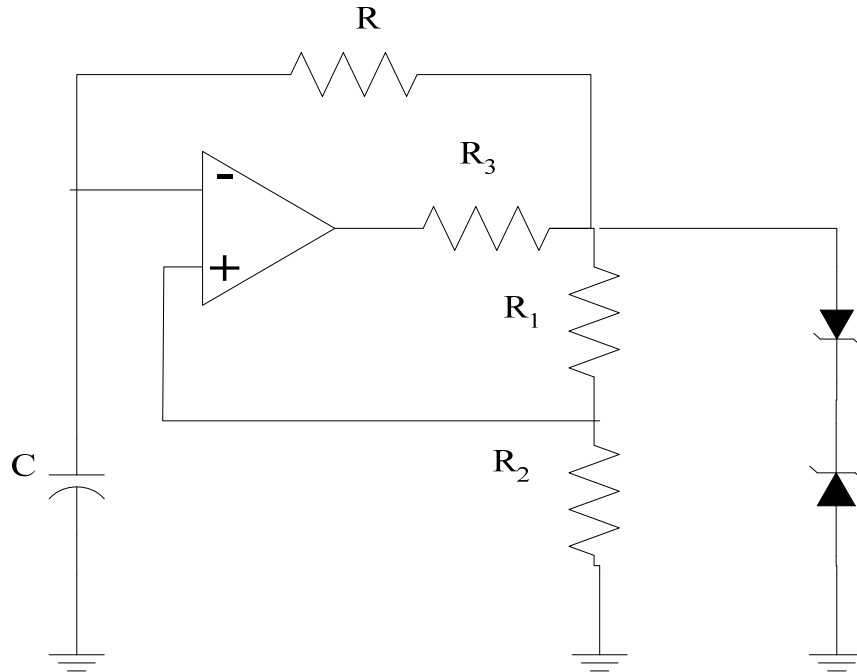where

$\bar{x}$ is the sample mean of the measured value.

Figure 1   Square-Wave Oscillator

If α is 0.05 then there is 90% confidence that the value of all possible circuits will be within the indicated range about the sample mean. When the sample size n is smaller than 30 or the population standard deviation, σ is not known the sample standard deviation, s can be substituted and the normal distribution must be replaced with the Student-T distribution [2, pp. 174-7].

Interpretation of the term $z(\alpha/2)$ and $t(\alpha/2, n-2)$ often leads to a lot of confusion. It is the $z-$ or $t-$ score value that is necessary so that $(1-\alpha)$ of the area under the probability distribution is inside the desired range. This is often referred to, in part, as a two-tailed test. Figure 2 shows this graphically.
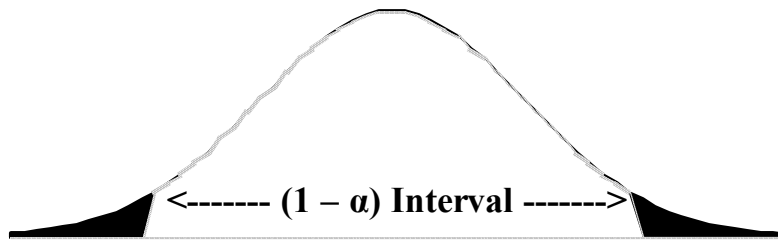
<----- (1 − α) Interval ----->

Figure 2 $(1-\alpha)$ Confidence Interval

$z(\alpha/2)$ is the value so that when the normal distribution is integrated from $-z(\alpha/2)$ to $+z(\alpha/2)$ that the resulting area is $(1-\alpha)$. This integration is summarized by the following equation.

$$(1-\alpha) = \int_{-z(\alpha/2)}^{z(\alpha/2)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

The analysis is similar for the Student-T distribution and $t(\alpha/2, n-2)$.

The problem is that there is no closed-form solution to the integration of either the normal distribution or the Student-T distribution. These are typically handled via look-up tables of the cumulative distributions from $-\infty$ to z [2, pp. 513-14].

In the case of a confidence interval computation generally everything is known except for $z(\alpha/2)$. However, in the case of a circuit, that analysis can be reversed to determine with what confidence the average circuit behavior will be in a given interval. In other words, suppose that the goal is to determine how likely it is that the square-wave generator for Figure 1 will behave with a frequency of 11 kHz. For example, suppose that suitable components have been chosen for R, C, R_1 and R_2 for a lot and that 25 circuits were built with the following results being measured:

> Number of Samples: $n = 25$
> Average Period: $\bar{x} = 0.995$ ms
> Standard Deviation: $\sigma = 0.02$ ms

What is the likelihood, with this batch of components, that any given oscillator will have its frequency within 5% of the target of 1 kHz? Rearranging the confidence interval formula and solving for $z(\alpha/2)$ yields the following two equations. One for the upper end for $\mu < 1.053$ ms and the other for the upper end for $\mu > 0.952$ ms^3.

$$z(\alpha_1/2) = \frac{(\mu_1 - \bar{x})}{\sigma} = \frac{(1.053m - 0.995m)}{0.02m} = 2.9$$

$$z(\alpha_2/2) = \frac{(\mu_2 - \bar{x})}{\sigma} = \frac{(0.952m - 0.995m)}{0.02m} = -2.15$$

The likelihood of this happening can then be determined by finding the total area under the normal distribution for the range in question

$$(1-\alpha)=\int\limits_{-2.15}^{2.9}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

QUESTIONS

1. There is no real way to know the population standard deviation, σ, so the more complex formula using the sample standard deviation, s, and the Student-T distribution would have to be used, especially when n is less than 30   30 [2, pp. 174-19].   The entire analysis is the same except the integral is now the more complicated.

$$(1-\alpha)=\int\limits_{-t(\alpha/2,m)}^{t(\alpha/2,m)}\left\{\frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{m\pi}\,\Gamma\left(\frac{m}{2}\right)}\left(1+\frac{x^2}{m}\right)^{-\left(\frac{m+1}{2}\right)}\right\}dx$$

where

$$\Gamma(1)=1$$
$$\Gamma(1/2)=\sqrt{\pi}$$
$$\Gamma(m)=(m-1)\Gamma(m-1)$$

Perform the computations using the Student-T distribution.

2. The more common way of using these statistics is to compute the actual confidence interval centered about the sample data. This means that α is known and $z(\alpha/2)$ or $t(\alpha/2,n-2)$ have to be computed. This can be done by repeatedly trying possible values for z or t and evaluating the integral. This is an excellent application for a root-finding algorithm. If an algorithm like bisection method of solving nonlinear equations were used, how would you pick the initial bounds for the x or t value?

3. Because there are actual formulas for the functional inside the integral and not just sampled points on the curve in question, it is possible to apply an interval halving approach and a Romberg extrapolation to solve these integrals. Set up the algorithm for doing this.

4. Normal and Student-T distribution tables like those in [2, pp. 513-4] are limited in their choices. Integrating these distributions from $-\infty$ can be used to generate a more complete table. To do this an algorithm must take advantage of the symmetry in the distributions since it is not possible to integrate numerically with an infinite bound. When the z or t value is less than zero simply report 0.5 minus the integral from 0 to z or t, and when the z or t value is greater than zero simply report 0.5 plus the integral from 0 to z or t. Set-up the algorithm for doing this.

5. In the analysis above it was assumed that the population of circuits was roughly normal, this should generally be tested before applying the assumption. There exist goodness-of-fit tests to see if the population is sufficiently normal. For more information, see [2, pp. 265-8].

References

[1] Millman, J., Micro-Electronics: Digital and Analog Circuits and Systems, McGraw Hill, 1979.

[2] Walpole, R. and Myers, R., Probability and Statistics for Engineers and Scientists, 2nd Ed., MacMillan, 1978.

Topic	INTEGRATION
Sub Topic	Physical Problem
Summary	Not all electrical components, especially off-the-shelf components match their nominal value. Variations in materials and manufacturing as well as operating conditions can affect their value. Suppose a circuit is designed such that it requires a specific component value, how confident can we be that the variation in the component value will result in acceptable circuit behavior? To solve this problem a probability density function will need to be integrated to determine the confidence interval.
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