

Chapter 08.00E

Physical Problem for Electrical Engineering Ordinary Differential Equations

Problem Statement

Small non-switching power supplies such as AC power bricks are typically built around a small transformer, rectifier, and voltage regulator as shown in Figure 1.

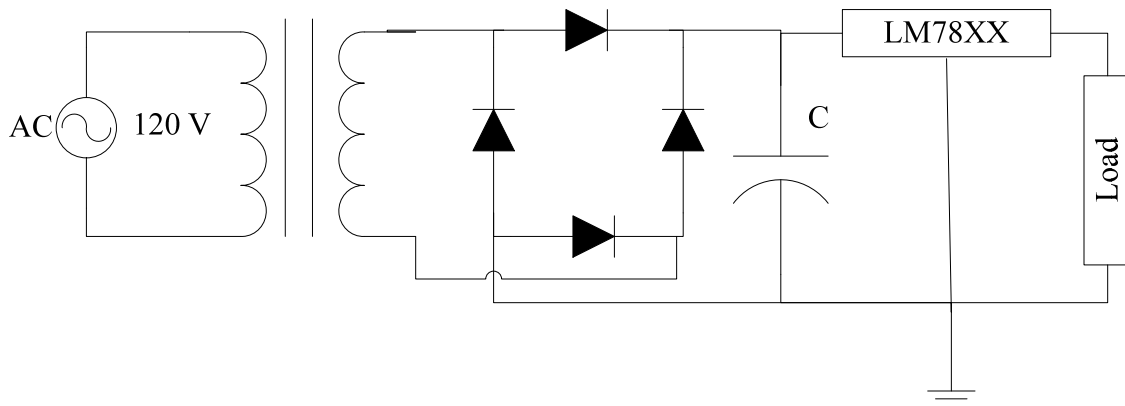


Figure 1. Full-Wave Rectified DC Power Supply

The transformer is used to reduce the AC voltage levels to a more reasonable range (e.g. 12 VAC or 18 VAC) and the bridge rectifier will take the negative half cycle of the AC waveform and convert it to a positive half cycle. A typical full-wave rectified waveform is shown in Figure 2. This waveform was generated assuming an ideal diode operating under no-load conditions and can be modeled with the following equation:

$$V_{full} = \max(|18 \times \cos(t) - 1.4|, 0)$$

For a half-wave rectified system built using only a single diode, the equation would become:

$$V_{half} = \max(18 \times \cos(t) - 0.7, 0)$$

and the waveform would look similar to Figure 2 except every other hump would be replaced with 0.

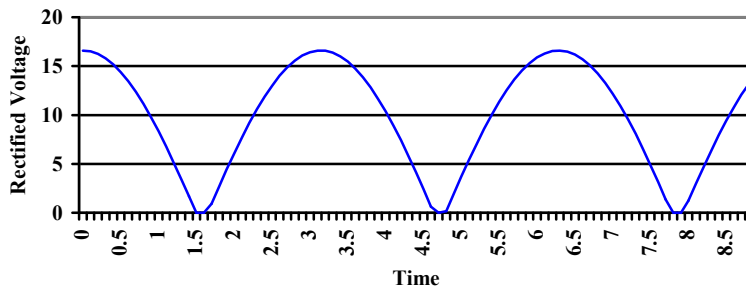


Figure 2 Full-Wave Rectified AC Signal.

This is clearly not suitable as a DC power source since the load is looking for a constant DC value. This is where the capacitor, C , and the LM78XX voltage regulator of Figure 1 become important. A typical voltage regulator requires that the voltage on the input pin maintain a certain margin above the regulated output voltage. This is typically in the range of 2.5 to 3 V and for the LM7805, which provides 5 VDC, it is 2.5 V [1]. This means that the minimum voltage at the input pin of the LM7805 must not drop below 7.5 V. Clearly, the waveform in Figure 2 does that regularly.

The task of the capacitor, C , in the power supply is to store up charge when the rectifier provides voltage in excess of the required minimum and then support the voltage when the regulated voltage is below the minimum by providing stored charge to the voltage regulator. Thus, the circuit has two distinct phases of operation.

Phase 1

When the rectified voltage is above the voltage across the capacitor V_c , then the rectifier provides current to the capacitor and the voltage regulator, and the capacitor charges. The ideal model of a diode will not be adequate here since we need to know how much current the diode can supply. A better model of the diode can be found from its VI characteristic such as the one shown in Figure 3. A model for this will be developed shortly.

Phase 2

When the rectified voltage drops below V_c current ceases to flow between the rectifier diodes and the capacitor. The capacitor then becomes the sole source of current to the voltage regulator by bleeding off its charge. This has the additional effect of continually reducing the voltage across the capacitor. As long as it does not fall below the required threshold, the regulator can successfully do its job.

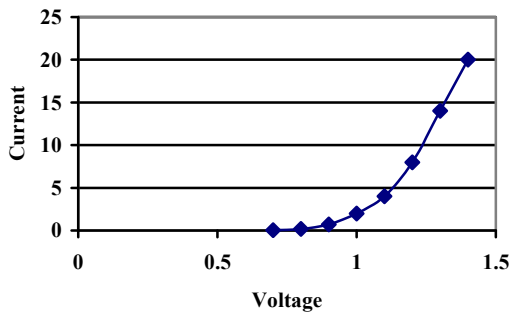


Figure 3 VI Characteristic of a 1N4001 Diode [2].

Diode Model

The diode of Figure 3 must be modeled to account properly for the current it can supply during the Phase 1 charging phase of the capacitor. It is reasonable to model a diode under these conditions as a voltage drop and a series resistance. This essentially treats the Figure 3 curve as piece-wise linear model and it can only be applied when the diode is on (that is, it does not properly model an under-biased or reverse-biased diode) [3, pp. 67-74]. This model is shown in Figure 4 and for the 1N4001 of Figure 3 suitable values are $V_d = 1.0\text{ V}$ and $R_d = 0.02\ \Omega$. Other diodes will have similar numbers.

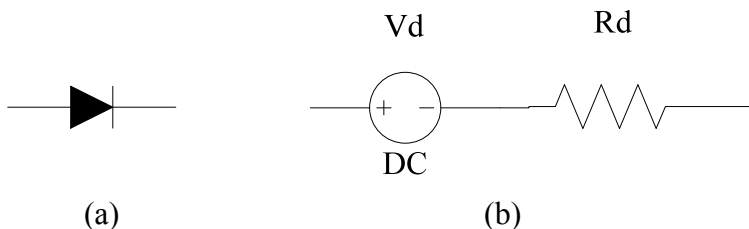


Figure 4 (a) Ideal Diode and (b) Piece-wise Forward Bias Model.

Load Model

The final piece to the picture is how to model the load represented by the voltage regulator. Rather than get into the details of the internal structure of the voltage regulator it is sufficient to model the worst-case scenario when choosing the appropriate capacitor. This is represented by the peak continuous current that the power-supply is expected to provide. For this problem, it will be assumed to be 100 mA.

Circuit Models

With the Phase 1 charging, and Phase 2 discharging, circuit models of the power supply circuit can now be built. They are shown in Figure 5. In both cases, the circuit must provide 100 mA to the load. It will be additionally assumed that a transformer will be chosen that provides sufficient AC power at 18 VAC running at 60 Hz.

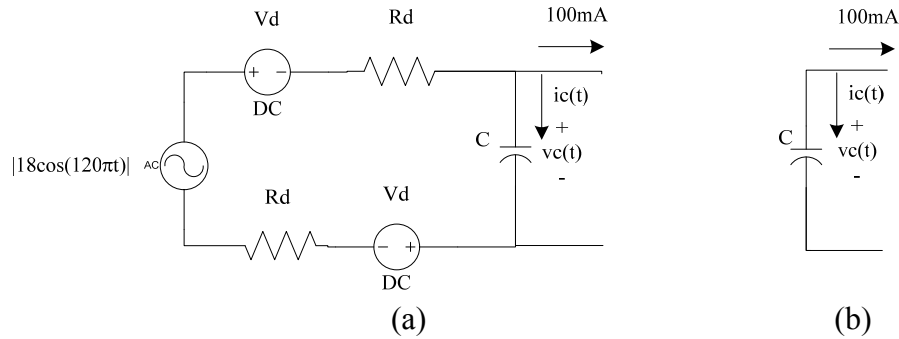


Figure 5 (a)Phase 1 (Charging) and (b) Phase 2 (Discharging) Circuit Models

Analysis

The analysis of these two circuits is rather straight-forward once the VI characteristic of the capacitor is substituted. In the time-domain, this is represented by the differential equation

$$i_c(t) = C \frac{dv_c(t)}{dt} \quad \text{subject to initial conditions on } v_c.$$

During the charging of Phase 1, the operative circuit equation is found by writing a Kirchoff's Current Law equation for the node at the top of the capacitor. Doing this yields

$$C \frac{dv_c(t)}{dt} + 100mA = \frac{|18 \cos(120\pi t)| - 2V_d - v_c(t)}{2R_d},$$

when

$$v_c(t) \leq |18 \cos(120\pi t)| - 2V_d.$$

During the discharging of Phase 2, the circuit is a bit simpler and the equation is

$$C \frac{dv_c(t)}{dt} + 100mA = 0,$$

when

$$v_c(t) > |18 \cos(120\pi t)| - 2V_d.$$

These are essentially the same differential equation it is just that the forcing function has two cases that must be monitored as the equation is solved. Substituting the previously derived values for V_d and R_d will yield the following equation in standard form.

$$\frac{dv_c(t)}{dt} = \frac{1}{C} \left\{ -0.1 + \max \left(\frac{|18 \cos(120\pi t)| - 2 - v_c(t)}{0.04}, 0 \right) \right\}.$$

Assuming that the capacitor is initially discharged (i.e. $v_c(0+) = 0$) it is simply a matter of trying various standard capacitors (e.g. 100, 150, 220, 330, 470, and 680 μF or other powers of 10 [4]) into the equation and simulating it for a few cycles of the AC source. Pick the smallest value where $v_c(t)$ never drops below the required 7.5 V.

QUESTIONS

1. Most capacitors are marked with a tolerance of $\pm 20\%$. How would your analysis handle the case that a 100 μF capacitor could be as high as 120 μF or as low as 80 μF ?

2. What effect would it have on the differential equation to use a more complex model for the diode?
3. Is it necessary to have a more detailed model of the voltage regulator? Alternatively, is it suitable to model it simply as a maximum current sink?
4. Modify the model to account for an AC brownout, that is, what would happen if the AC power source saw a temporary drop in voltage?

References

- [1] LM340/LM78XX Data Sheet, National Semiconductor, <http://www.national.com/ds/LM/LM340.pdf>, accessed May 11, 2005.
- [2] 1N4001 Data Sheet, Fairchild Semiconductor, <http://www.fairchildsemi.com/ds/1N/1N4007.pdf>, accessed May 12, 2005.
- [3] Malvino, A., Electronic Principles, Sixth Edition, Glencoe McGraw-Hill, 1999.
- [4] Standard Capacitor Values, RF Café, http://www.rfcafe.com/references/electrical/capacitor_values.htm, accessed May 12, 2005.

ORDINARY DIFFERENTIAL EQUATIONS

Topic	Ordinary Differential Equations
Summary	A rectifier-based power supply requires a capacitor to store power temporarily when the rectified waveform from the AC source drops below the target voltage. To size this capacitor properly, a first-order ordinary differential equation must be solved.
Major	Electrical Engineering
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Date	May 24,2005
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