Chapter 02.00A

Physical Problem for Differentiation
General Engineering

Problem Statement

A rocket is traveling vertically and expels fuel at a velocity of 2000 m/s at a consumption rate of 2100 kg/s . The initial mass of the rocket is 140,000 kg. If the rocket starts from rest, how can I calculate the acceleration of the rocket at 16 seconds?

Figure 1 A rocket launched into space[1]

[1] Source of rocket picture: NASA Langley Research Center, Office of Education, edu.larc.nasa.gov/pstp/

Solution

If

m_0 = initial mass of rocket at $t = 0$ (kg),

q = rate at which fuel is expelled (kg/s),

u = velocity at which the fuel is being expelled (m/s),

then since the fuel is expelled from the rocket, the mass of the rocket keeps decreasing with time. The mass of the rocket, m at anytime, t is

$$m = m_o - qt$$

The forces on the rocket at any time are found by applying Newton's second law of motion. Then

$$\sum F = ma$$
$$uq - mg = ma$$
$$uq - (m_o - qt)g = (m_o - qt)a$$

where

g = acceleration due to gravity $\left(\text{m/s}^2\right)$

$$a = \frac{uq}{m_o - qt} - g$$

$$\frac{dv}{dt} = \frac{uq}{m_o - qt} - g \qquad (1)$$

$$v = -u\log_e\left(m_0 - qt\right) - gt + C$$

Since the rocket starts from rest

$$v = 0 \text{ at } t = 0$$

$$0 = -u\log_e\left(m_o\right) + C$$

$$C = u\log_e\left(m_o\right)$$

Hence

$$v = -u\log_e\left(m_o - qt\right) - gt + u\log_e\left(m_o\right)$$

$$\frac{dx}{dt} = u\log_e\left(\frac{m_o}{m_o - qt}\right) - gt \qquad (2)$$

$$u = 2{,}000 \text{ m/s}$$

$$m_o = 140{,}000 \text{ kg}$$

$$q = 2100 \text{ kg/s}$$

$$g = 9.8 \text{ m/s}^2$$

$$t_0 = 0\text{s}$$

$$t_1 = 30\text{s}$$

$$v = 2000\log_e\left(\frac{14\times10^4}{14\times10^4 - 2100t}\right) - 9.8t\,.$$

Can you numerically find the acceleration at $t = 16\,s$? You may say that we do not need numerical or analytical differentiation to calculate the acceleration, as equation (1) directly gives us the acceleration of the rocket at any time. True! We are just doing this as an exercise to illustrate numerical differentiation and we have a true expression of acceleration readily available for comparison with the numerical results.

Questions

1. Find the acceleration of the rocket at $\text{t} = 16\text{s}$.
2. Use different numerical differentiation techniques to find the acceleration at $\text{t} = 16\text{s}$. Compare these results with the exact answer.

DIFFERENTIATION	
Topic	Physical problem
Summary	A physical problem of find the acceleration of a rocket.
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