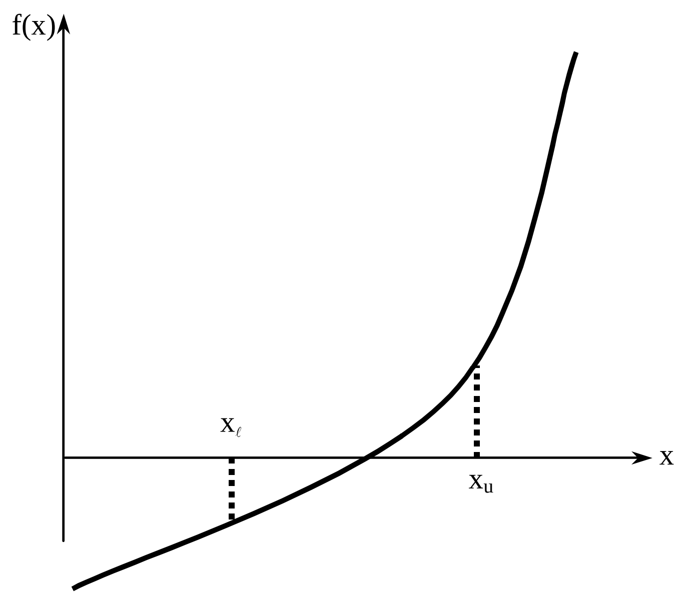
Roots of a Nonlinear Equation

Topic: Bisection Method

Major: General Engineering

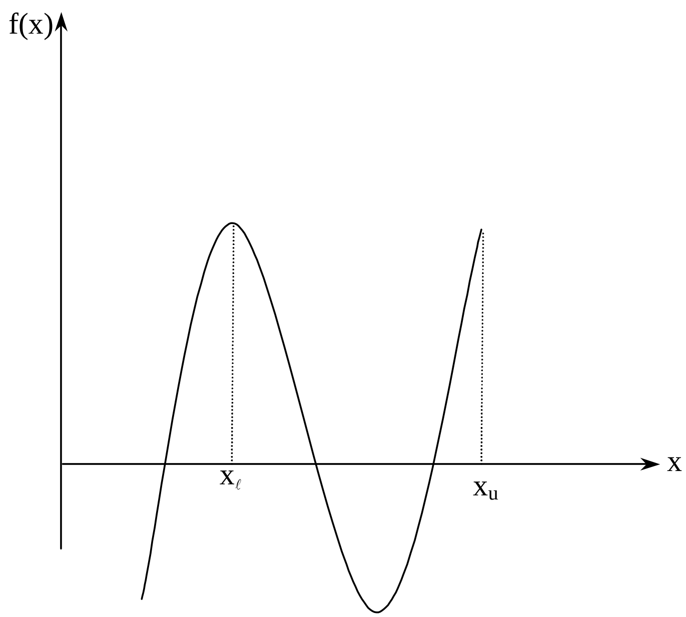
Basis of Bisection Method

Theorem: An equation f(x)=0, where f(x) is a real continuous function, has at least one root between x_l and x_u if $f(x_l)\ f(x_u) < 0$.



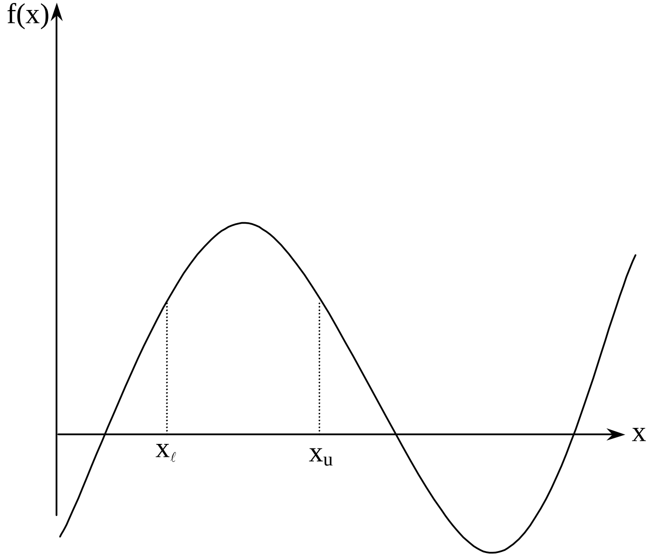
Theorem

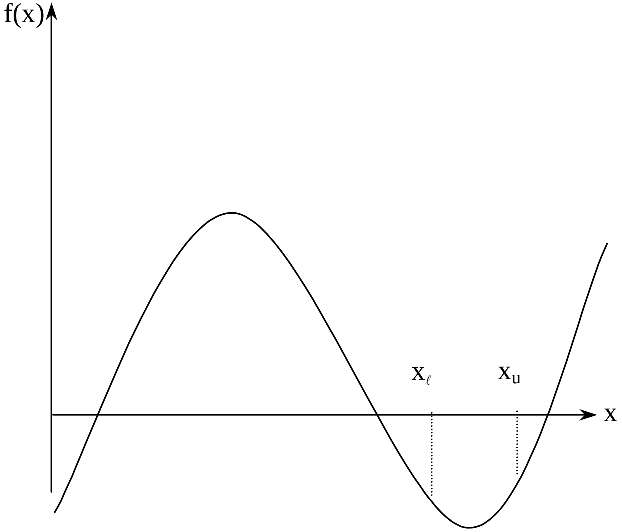
If function f(x) in f(x)=0 does not change sign between two points, roots may still exist between the two points.



Theorem

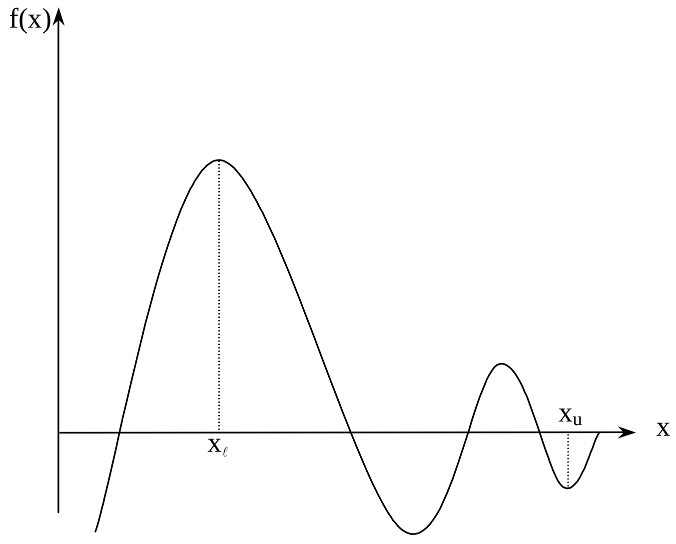
If the function f(x) in f(x)=0 does not change sign between two points, there may not be any roots between the two points.





Theorem

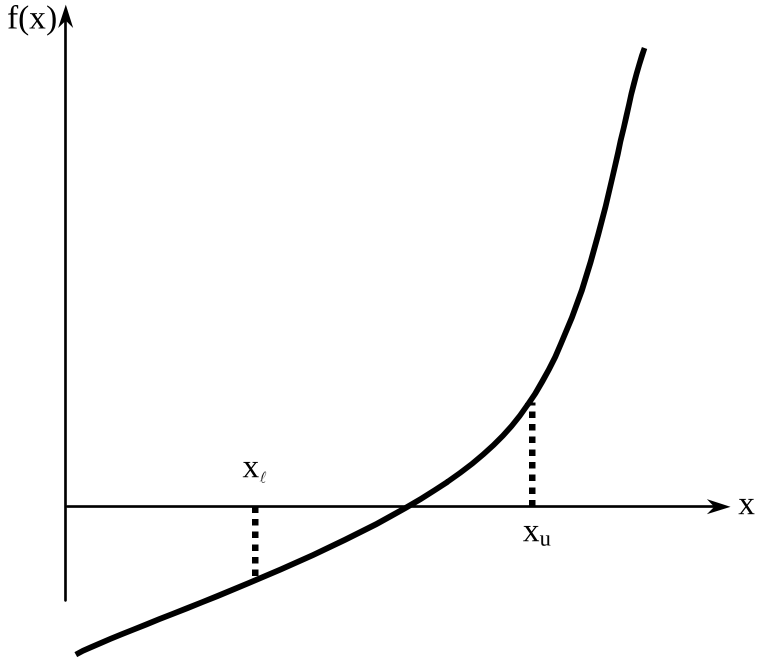
If the function f(x) in f(x)=0 changes sign between two points, more than one root may exist between the two points.



Algorithm for Bisection Method

Step 1

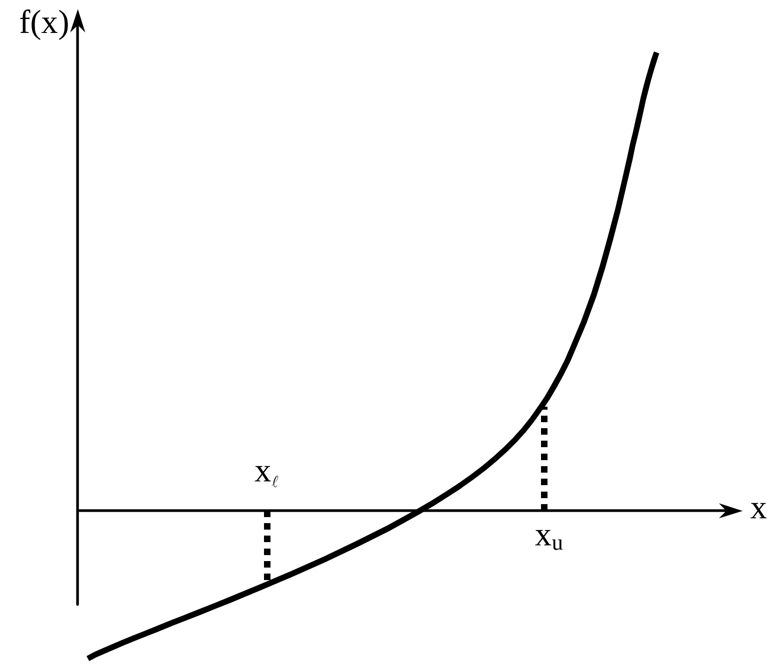
- Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)\, f(x_u) < 0$, or in other words, $f(x)$ changes sign between x_ℓ and x_u.



Step 2

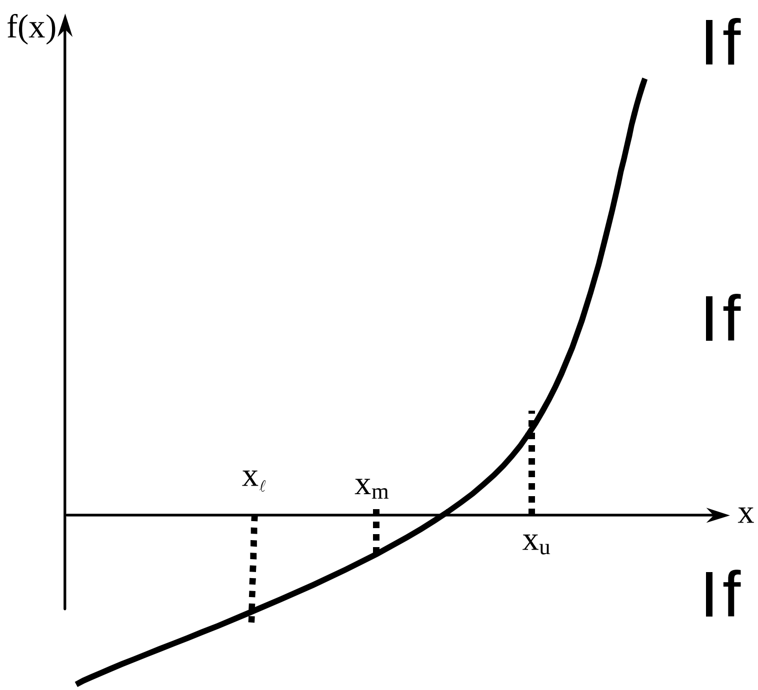
Estimate the root, x_m of the equation f (x) = 0 as the mid-point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$



Step 3

Now check the following



If $f(x_\ell)\ f(x_m) < 0$, then the root lies between $x_{☼}$ and x_m; then $x_\ell = x_\ell$; $x_u = x_m$.

If $f(x_\ell\ )\ f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\ell = x_m$; $x_u = x_u$.

If $f(x_\ell)\ f(x_m) = 0$; then the root is x_m. Stop the algorithm if this is true.

Step 4

New estimate

$$x_m = \frac{x_\ell + x_u}{2}$$

Absolute Relative Approximate Error

$$|\epsilon_a| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100$$

x_m^{old} = previous estimate of root

x_m^{new} = current estimate of root

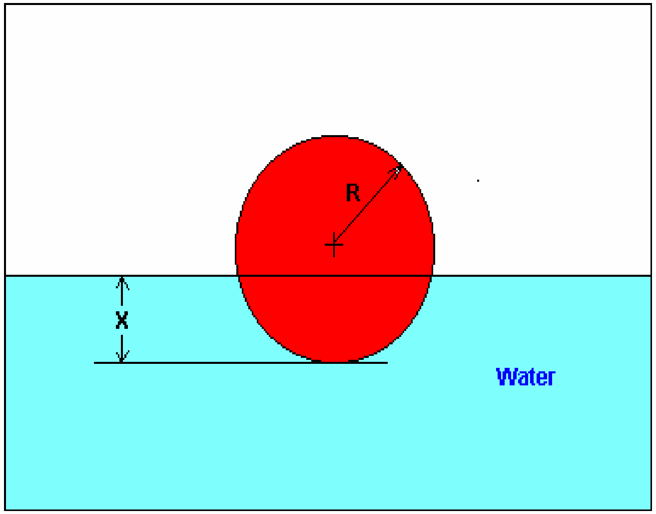
Step 5

Check if absolute relative approximate error is less than prespecified tolerance or if maximum number of iterations is reached.

- Yes → Stop
- No → Using the new upper and lower guesses from Step 3, go to Step 2.

Example

- You are working for 'DOWN THE TOILET COMPANY' that makes floats for ABC commodes. The ball has a specific gravity of 0.6 and has a radius of 1 cm. You are asked to find the distance to which the ball will get submerged when floating in water.

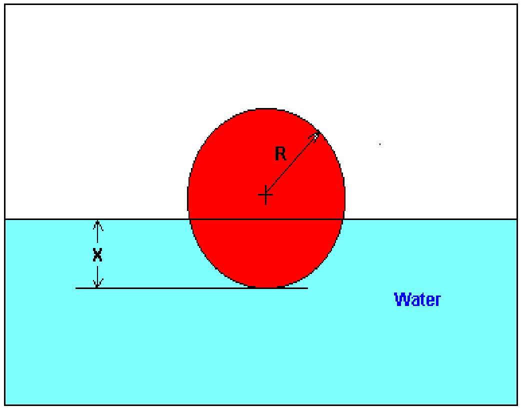


Solution

The equation that gives the depth 'x' to which the ball is submerged under water is given by

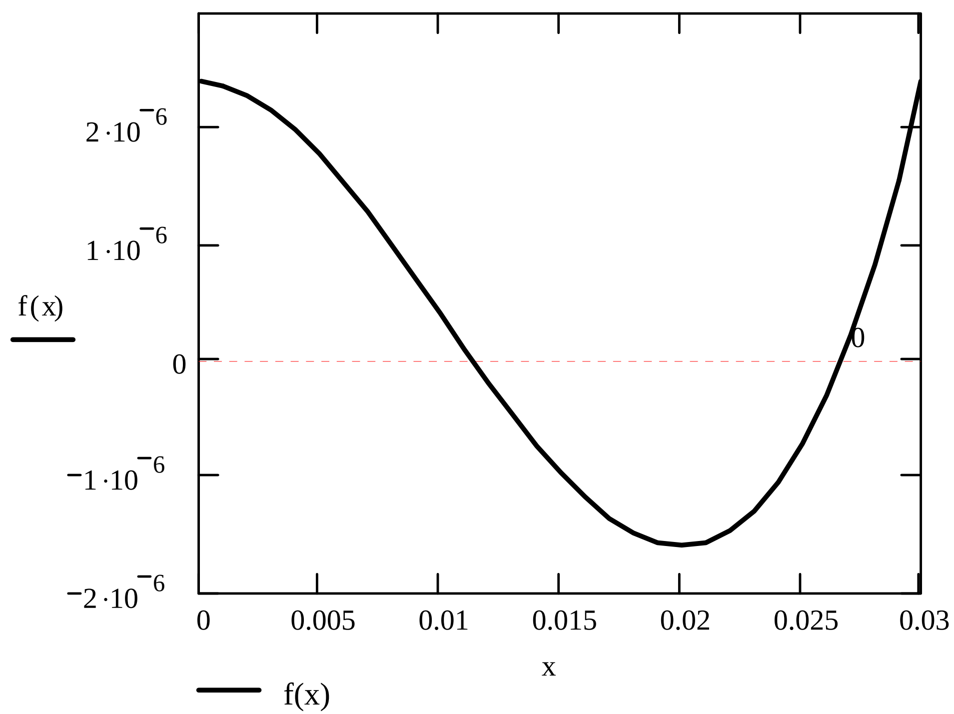
$$x^3 - 0.03x^2 + 2.4 \times 10^{-6} = 0$$

Use the bisection method of finding roots of equations to find the depth 'x' to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation.

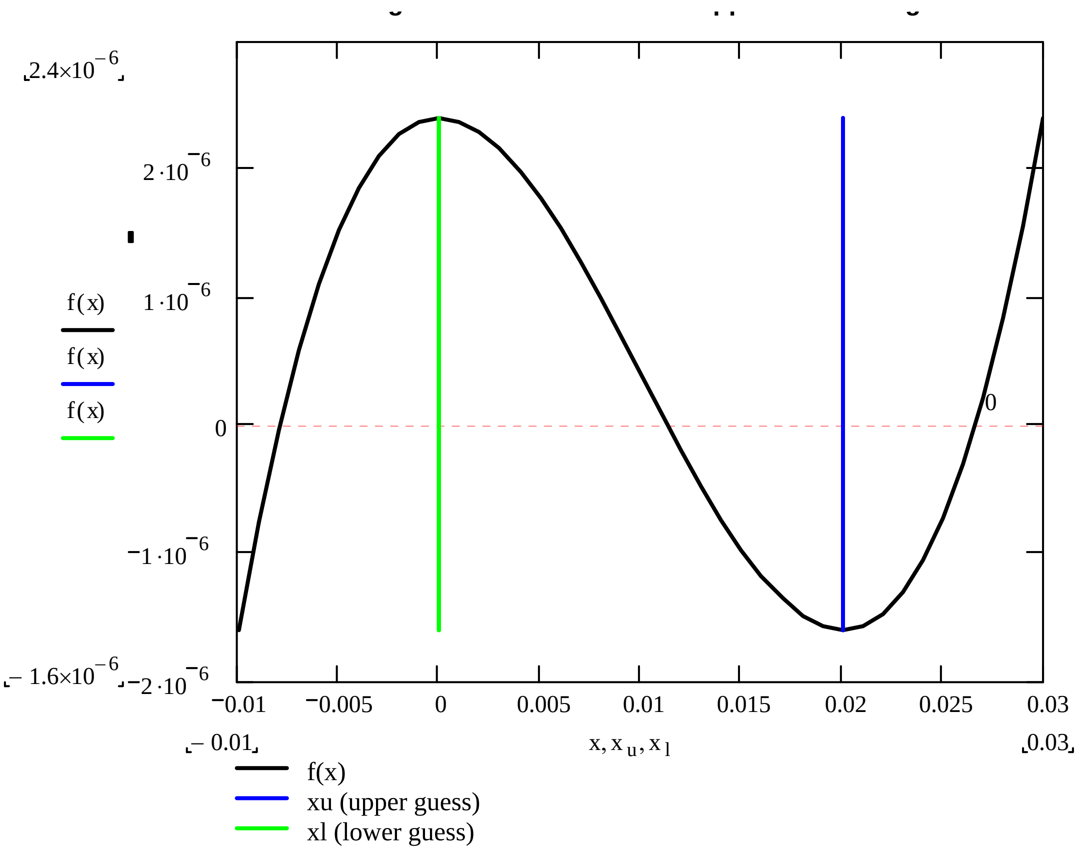


Graph of function f(x)

$$f(x) = x^3 - 0.03x^2 + 2.4\text{x}10^{-6} = 0$$



Checking if the bracket is valid



Choose the bracket

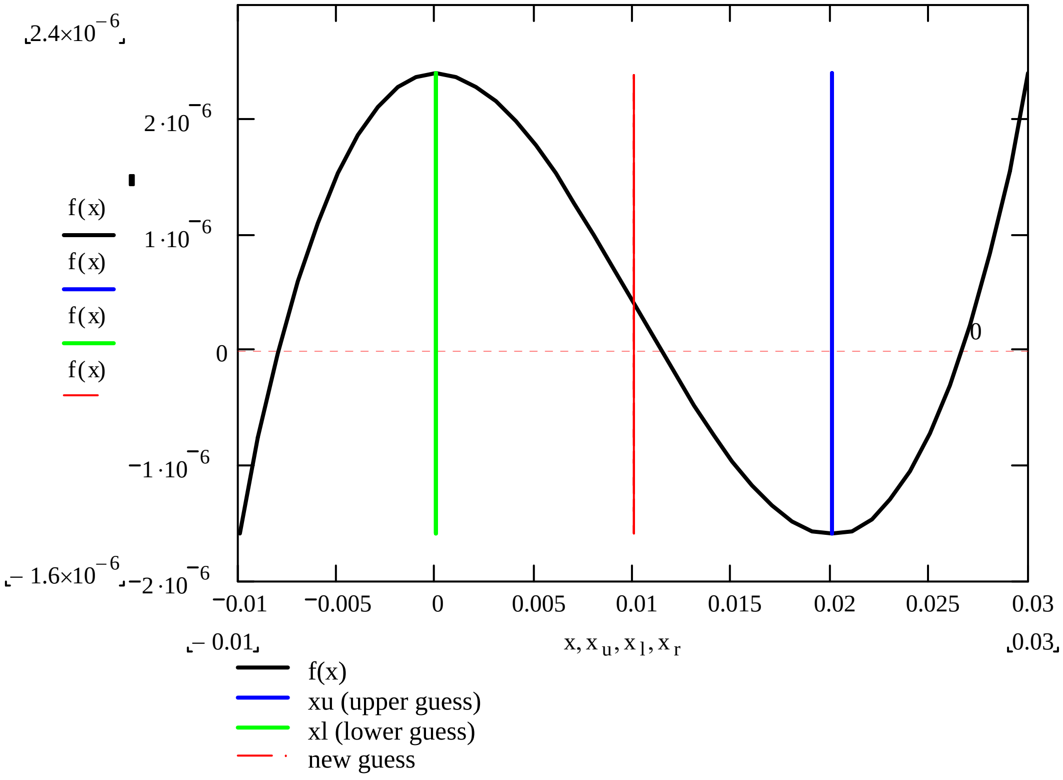
$$x_\ell = 0.00$$

$$x_u = 0.02$$

$$f(0.0) = 2.4\text{x}10^{-6}$$

$$f(0.02) = -1.6\text{x}10^{-6}$$

Iteration #1



$$x_\ell = 0, x_u = 0.02$$

$$x_m = \frac{0+0.02}{2} = 0.01$$

$$f(0) = 2.4\text{x}10^{-6}$$

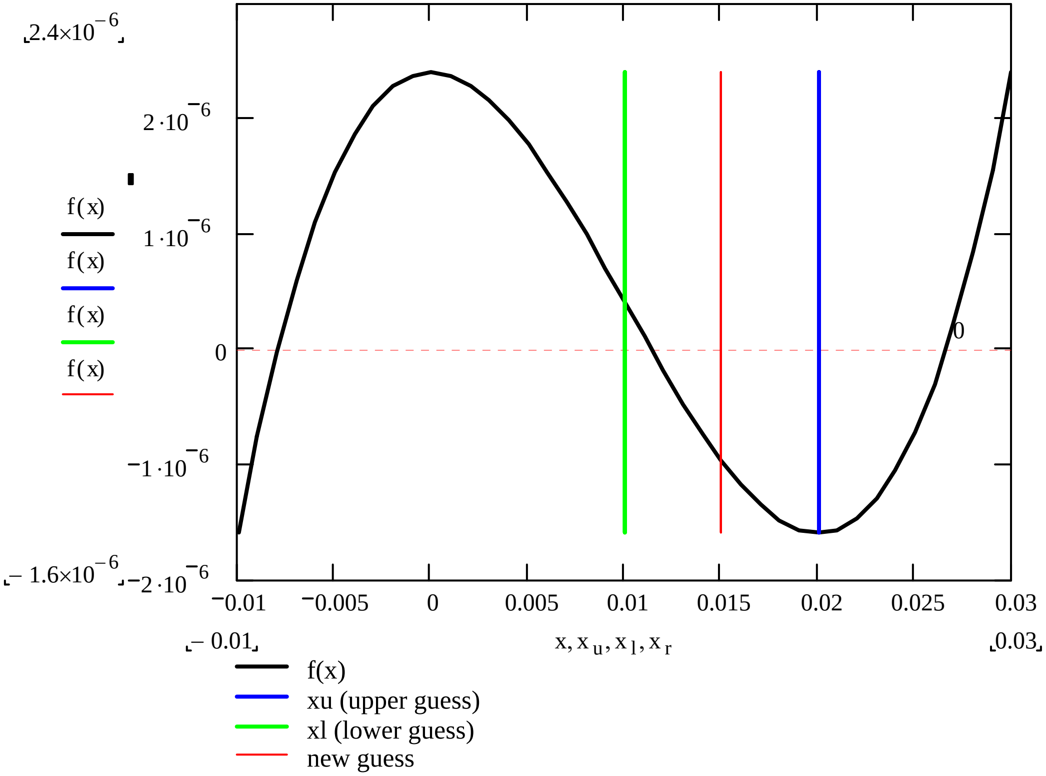
$$f(0.02) = -1.6\text{x}10^{-6}$$

$$f(0.01) = 4\text{x}10^{-7}$$

$$x_\ell = 0.01$$

$$x_u = 0.02$$

Iteration #2



$$x_\ell = 0.01, x_u = 0.02$$

$$x_m = \frac{0.01 + 0.02}{2} = 0.015$$

$$|\in_a| = 33.33\%$$

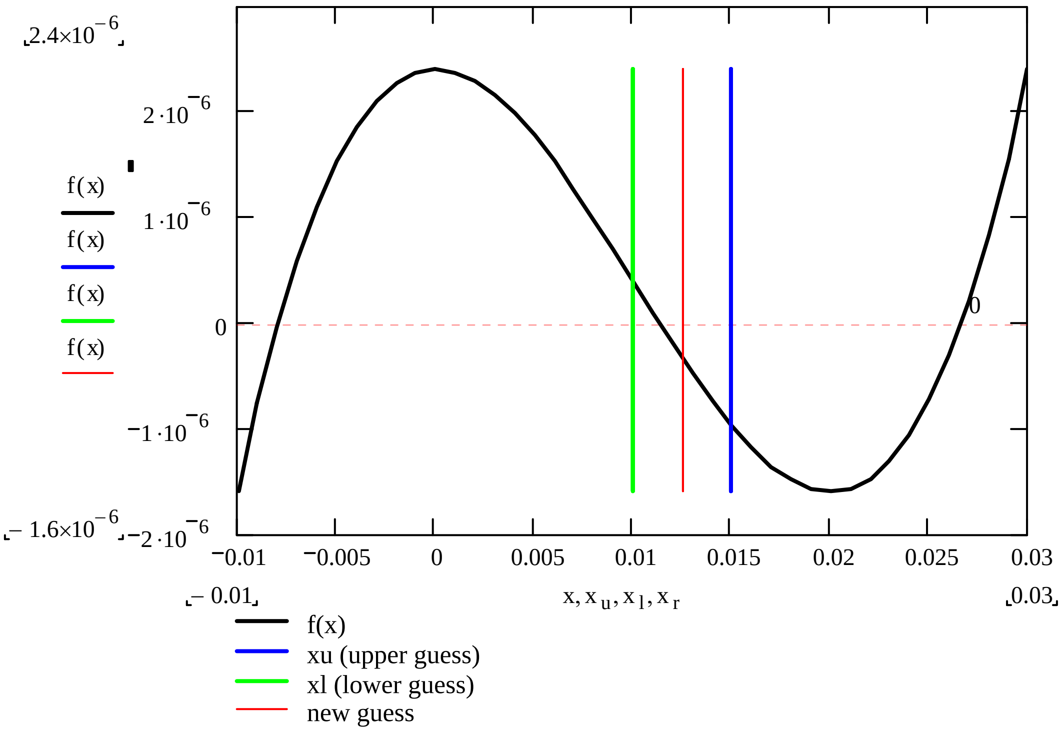
$$f(0.01) = 4\text{x}10^{-7}$$

$$f(0.02) = -1.6\text{x}10^{-7}$$

$$f(0.015) = -9.75\text{x}10^{-7}$$

$$x_\ell = 0.01, x_u = 0.015$$

Iteration #3



$$x_\ell = 0.01, x_u = 0.015$$

$$x_m = \frac{0.01+0.015}{2} = 0.0125$$

$$|\in_a| = 20\%$$

$$f(0.01) = 4\text{x}10^{-7}$$

$$f(0.015) = -9.75\text{x}10^{-7}$$

$$f(0.0125) = -3.3438\text{x}10^{-7}$$

Convergence

Table 1: Root of f(x)=0 as function of number of iterations for bisection method.

Iteration	x_ℓ	x_u	x_m	$\|\in_a\|$ %	$f(x_m)$
1	0.00000	0.0200	0.01000	----------	4.000×10^{-7}
2	0.01000	0.0200	0.01500	33.33	-9.750×10^{-7}
3	0.01000	0.0150	0.0125	20.00	-3.3438×10^{-7}
4	0.01000	0.0125	0.01125	11.11	2.6953×10^{-8}
5	0.01125	0.0125	0.011875	5.263	-1.5591×10^{-7}
6	0.01125	0.011875	0.011563	2.702	-6.4935×10^{-8}
7	0.01125	0.011563	0.011406	1.369	-1.9094×10^{-8}
8	0.01125	0.011406	0.011328	0.6896	3.9052×10^{-8}
9	0.011328	0.011406	0.11367	0.3436	-7.6007×10^{-9}
10	0.011328	0.011367	0.011348	0.1721	-1.8493×10^{-9}

Advantages

- Always convergent
- The root bracket gets halved with each iteration - guaranteed.

Drawbacks

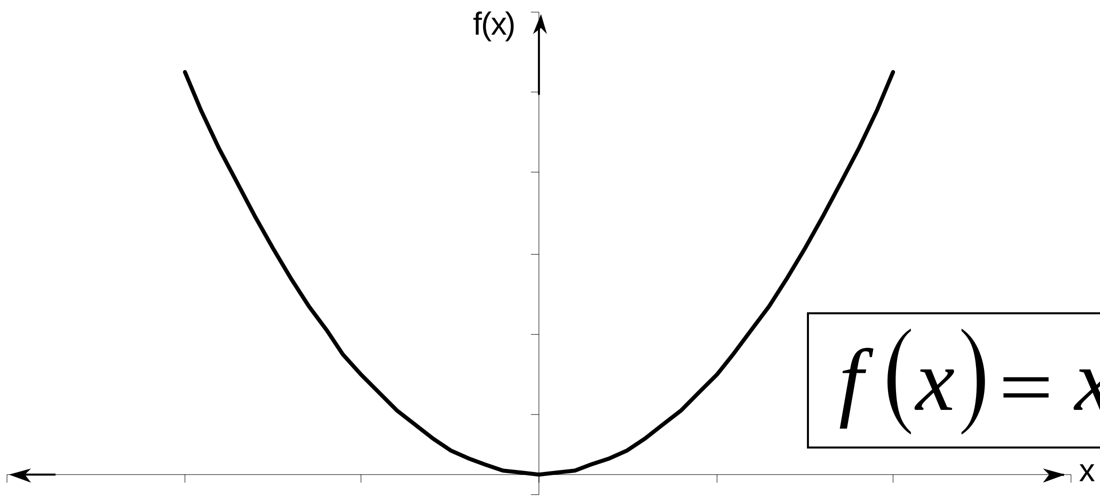
- Slow convergence

Drawbacks (continued)

- If one of the initial guesses is close to the root, the convergence is slower

Drawbacks (continued)

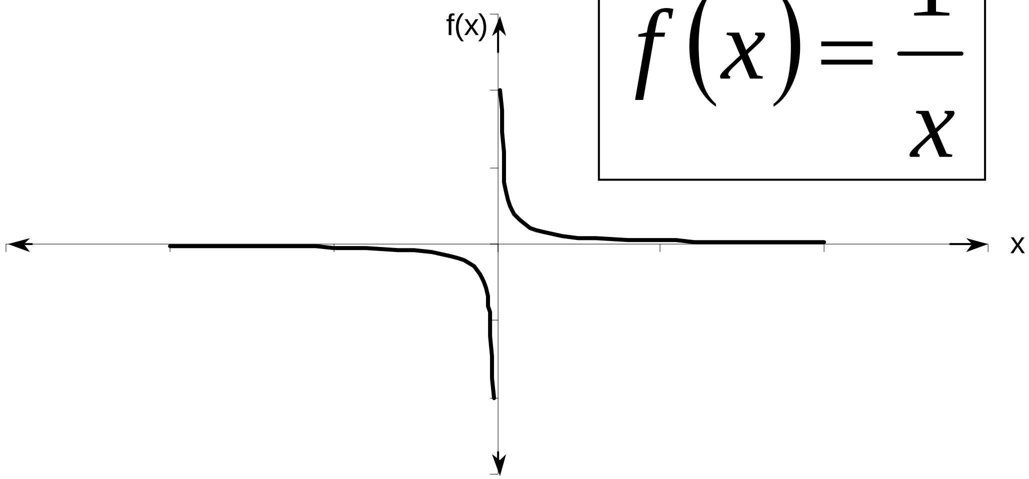
- If a function f(x) is such that it just touches the x-axis it will be unable to find the lower and upper guesses.



$$f(x) = x^2$$

Drawbacks (continued)

- Function changes sign but root does not exist



$$f(x) = \frac{1}{x}$$