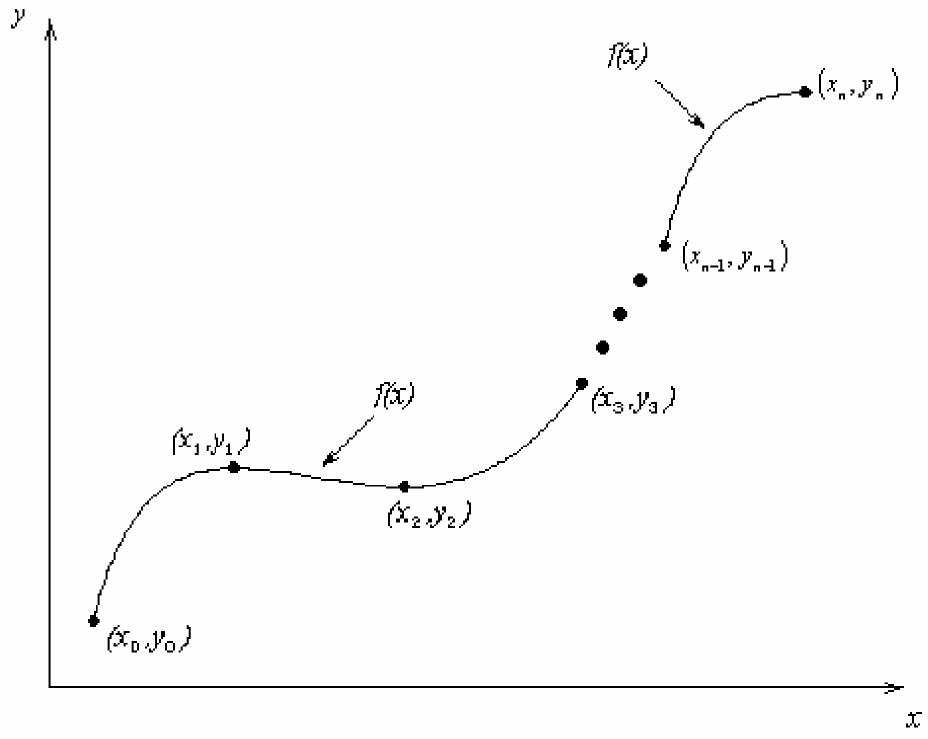
Interpolation

Topic: Direct Method

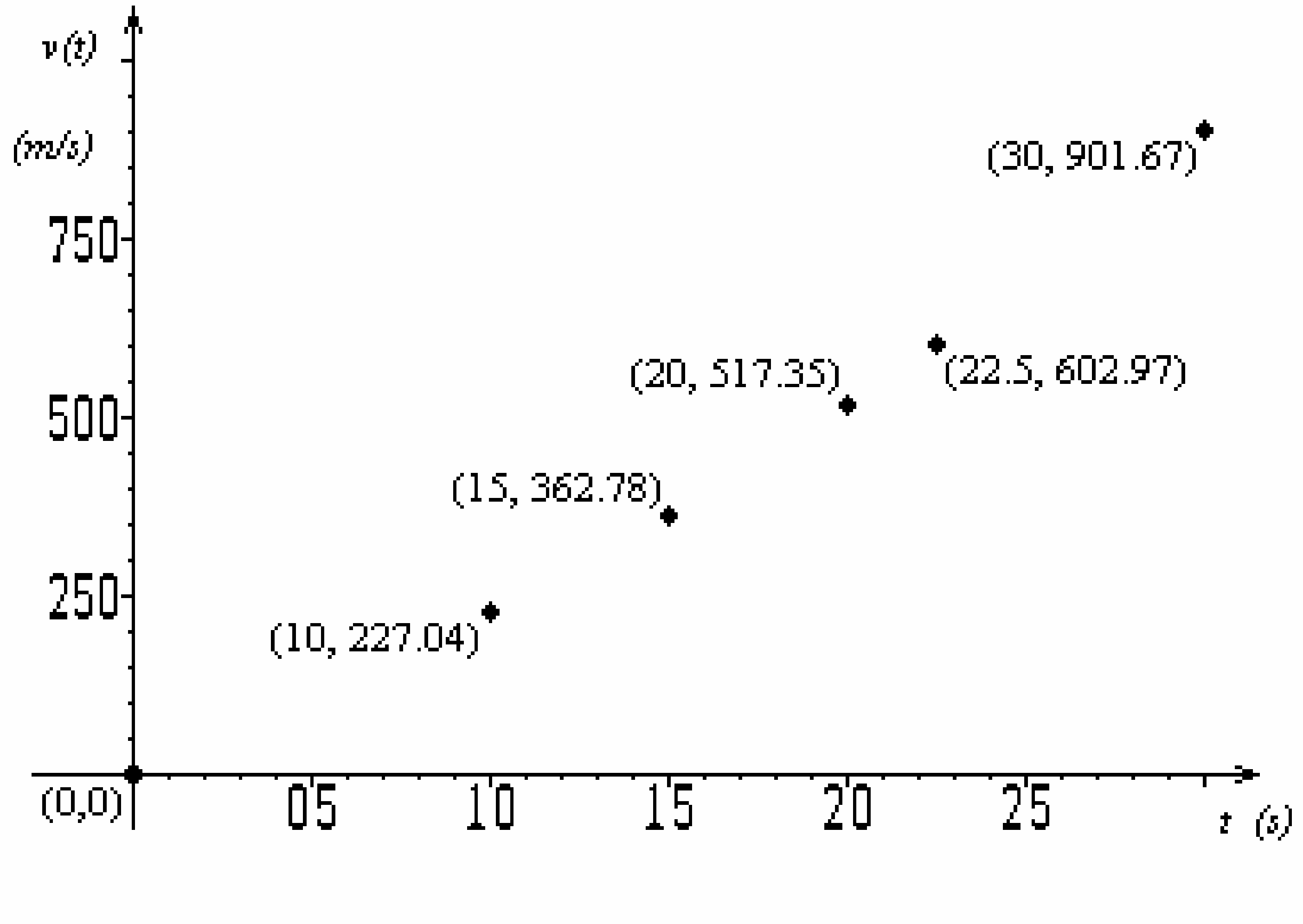
Major: General

What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Velocity vs Time



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate

Direct Method

Given 'n+1' data points (x_0,y_0), (x_1,y_1),............ (x_n,y_n), pass a polynomial of order 'n' through the data as given below:

$$y = a_0 + a_1 x + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_n x^n.$$

where a_0, a_1,................. a_n are real constants.

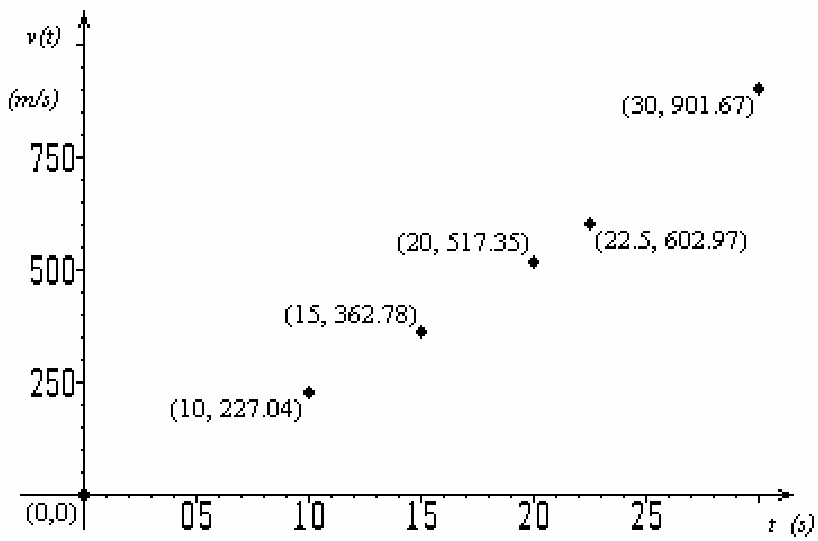
- Set up 'n+1' equations to find 'n+1' constants.
- To find the value 'y' at a given value of 'x', simply substitute the value of 'x' in the above polynomial.

Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using the direct method for linear interpolation.

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Velocity as a function of time



Velocity vs. time data for the rocket example

Linear Interpolation

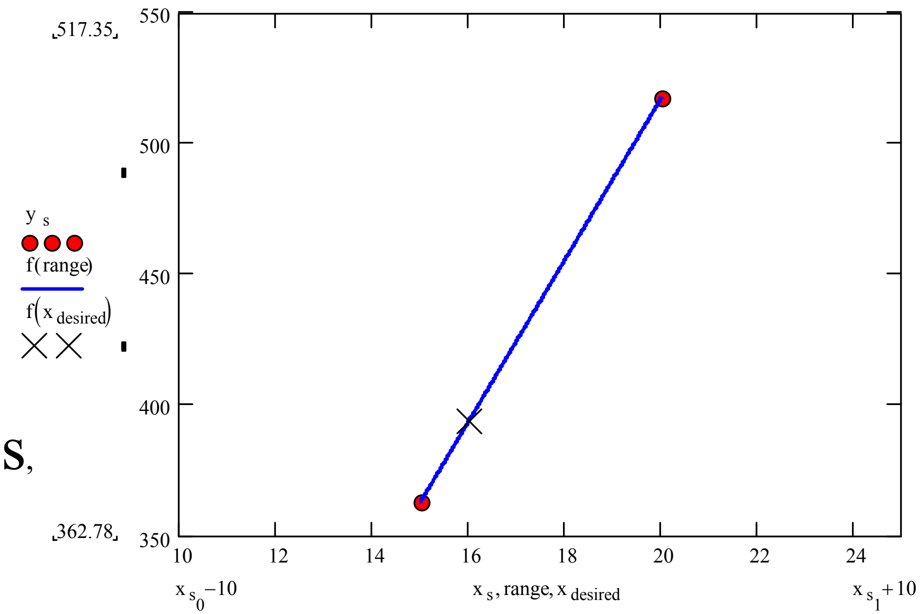
$$v(t) = a_0 + a_1 t$$

$$v(15) = a_0 + a_1(15) = 362.78$$

$$v(20) = a_0 + a_1(20) = 517.35$$

Solving the above two equations gives,

$$a_0 = -100.91 \qquad a_1 = 30.913$$



Hence

$$v(t) = -100.91 + 30.913t,\ \ 15 \le t \le 20.$$

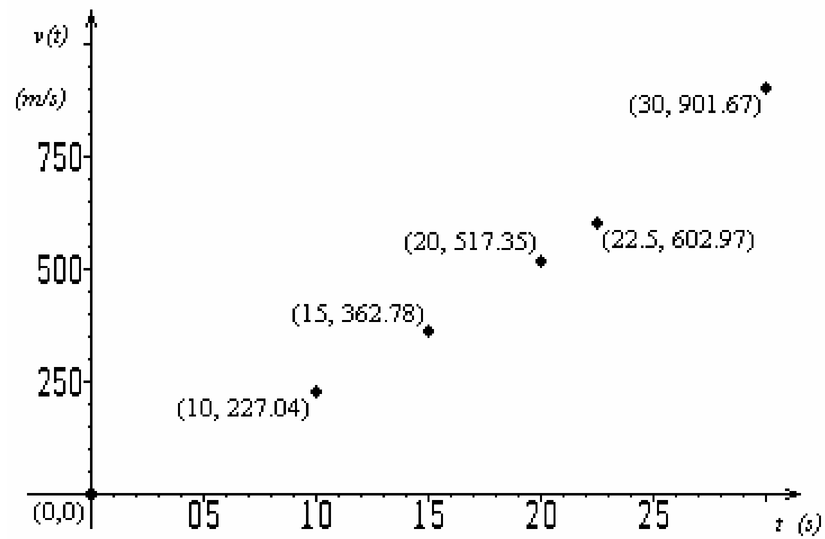
$$v(16) = -100.91 + 30.913(16) = 393.7\, m/s$$

Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using the direct method for quadratic interpolation.

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Velocity as a function of time



Velocity vs. time data for the rocket example

Quadratic Interpolation

$$v(t) = a_0 + a_1 t + a_2 t^2$$

$$v(10) = a_0 + a_1(10) + a_2(10)^2 = 227.04$$

$$v(15) = a_0 + a_1(15) + a_2(15)^2 = 362.78$$

$$v(20) = a_0 + a_1(20) + a_2(20)^2 = 517.35$$

Solving the above three equations gives

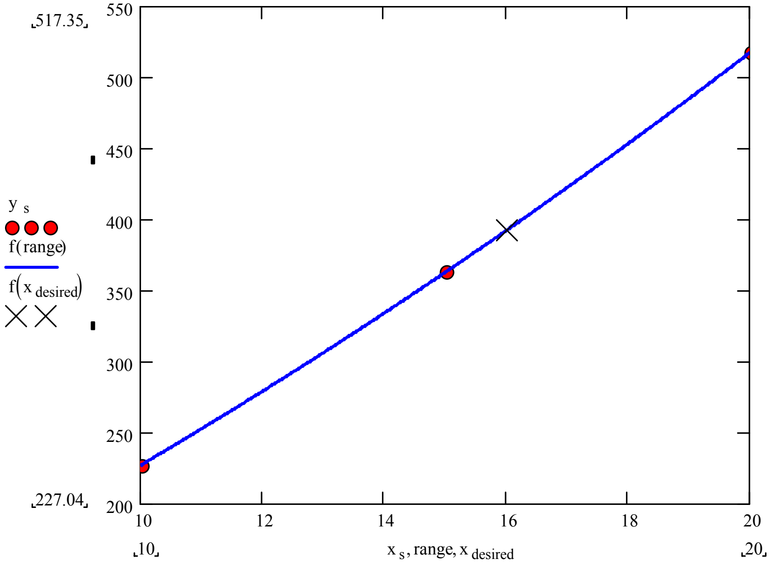
$$a_0 = 12.001 \quad a_1 = 17.740 \quad a_2 = 0.37637$$

Quadratic Interpolation (contd)

$$v(t) = 12.001 + 17.740t + 0.37637t^2,\ 10 \le t \le 20$$

$$v(16) = 12.001 + 17.740(16) + 0.37637(16)^2$$

$$= 392.19 m/s$$



The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{392.19 - 393.70}{392.19}\right| \times 100$$

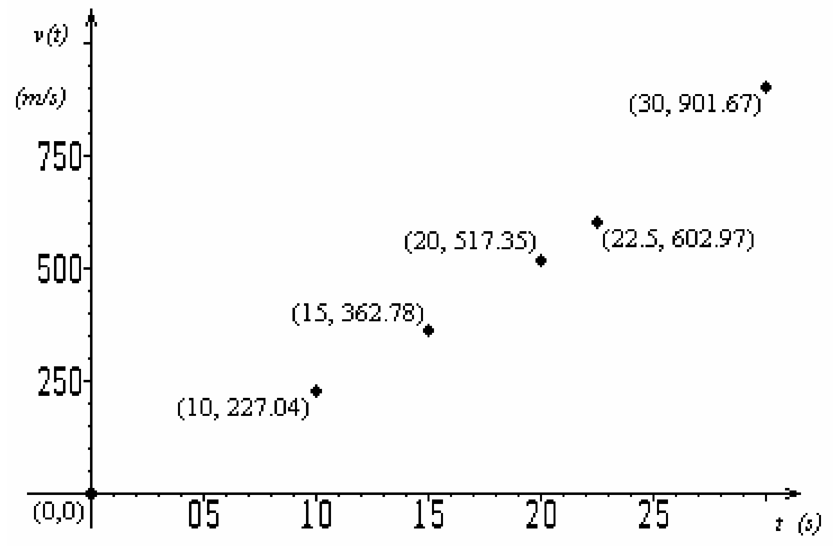
$$= 0.38502\%$$

Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using the direct method for cubic interpolation.

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Velocity as a function of time



Velocity vs. time data for the rocket example

Cubic Interpolation

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$v(10) = 227.04 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3$$

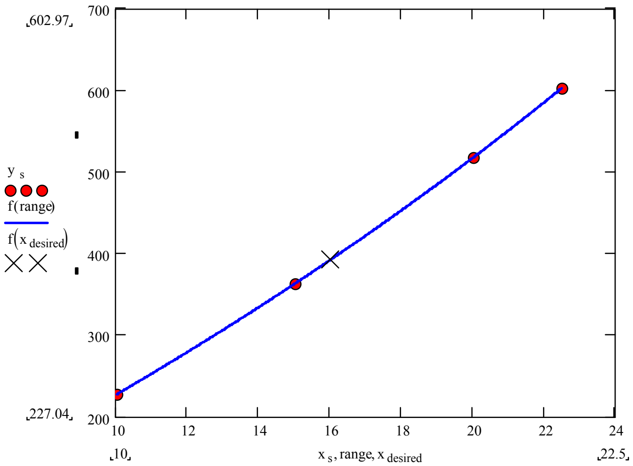
$$v(15) = 362.78 = a_0 + a_1(15) + a_2(15)^2 + a_3(15)^3$$

$$v(20) = 517.35 = a_0 + a_1(20) + a_2(20)^2 + a_3(20)^3$$

$$v(22.5) = 602.97 = a_0 + a_1(22.5) + a_2(22.5)^2 + a_3(22.5)^3$$

$a_0 = -4.3810$ $\quad a_1 = 21.289$ $\quad a_2 = 0.13065$ $\quad a_3 = 0.0054606$

Cubic Interpolation (contd)



$$v(t) = -4.3810 + 21.289t + 0.13064t^2 + 0.0054606t^3, \quad 10 \le t \le 22.5$$

$$v(16) = -4.3810 + 21.289(16) + 0.13064(16)^2 + 0.0054606(16)^3 = 392.06\,m/s$$

The absolute percentage relative approximate error, $|\in_a|$ between second and third order polynomial is

$$|\in_a| = \left|\frac{392.06 - 392.19}{392.06}\right| \times 100$$

$$= 0.033427\%$$

Comparison Table

Order of Polynomial	1	2	3
v(t=16) m/s	393.69	392.19	392.06
Absolute Relative Approximate Error	----------	0.38502 %	0.033427 %

Distance from Velocity Profile

Find the distance covered by the rocket from t=11s to t=16s ?

$$v(t) = -4.3810 + 21.289t + 0.13064t^2 + 0.0054606t^3, \quad 10 \le t \le 22.5$$

$$s(16) - s(11) = \int_{11}^{16} v(t)\,dt$$

$$\approx \int_{11}^{16} \left(-4.3810 + 21.289t + 0.13065t^2 + 0.0054606t^3\right)dt$$

$$= \left[-4.3810t + 21.289\frac{t^2}{2} + 0.13065\frac{t^3}{3} + 0.0054606\frac{t^4}{4}\right]_{11}^{16}$$

$$= 1605\,m$$

Acceleration from Velocity Profile

Find the acceleration of the rocket at t=16s given that

$$v(t) = -4.3810 + 21.289t + 0.13065t^2 + 0.0054606t^3, 10 \le t \le 22.5$$

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}\left(-4.3810 + 21.289t + 0.13064t^2 + 0.0054606t^3\right)$$

$$= 21.289 + 0.26130t + 0.016382t^2, \quad 10 \le t \le 22.5$$

$$a(16) = 21.289 + 0.26130(16) + 0.016382(16)^2$$

$$= 29.664 m/s^2$$