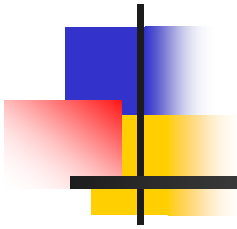




Interpolation



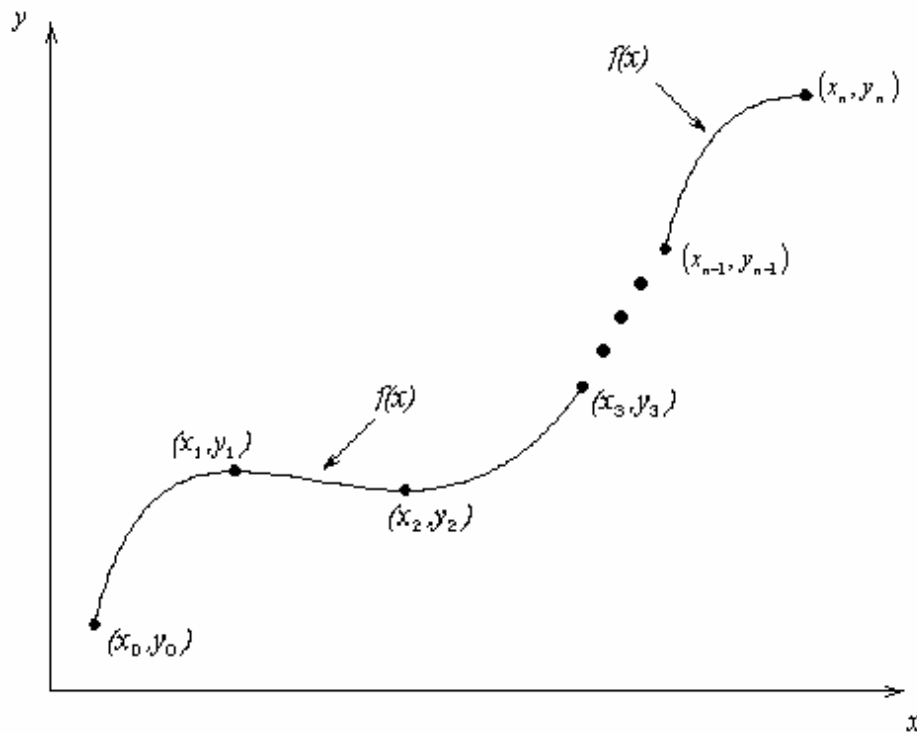
Topic: Spline Interpolation Method

Major: General



What is Interpolation ?

Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, find the value of 'y' at a value of 'x' that is not given.





Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Why Splines ?

$$f(x) = \frac{1}{1 + 25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

x	$y = \frac{1}{1 + 25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

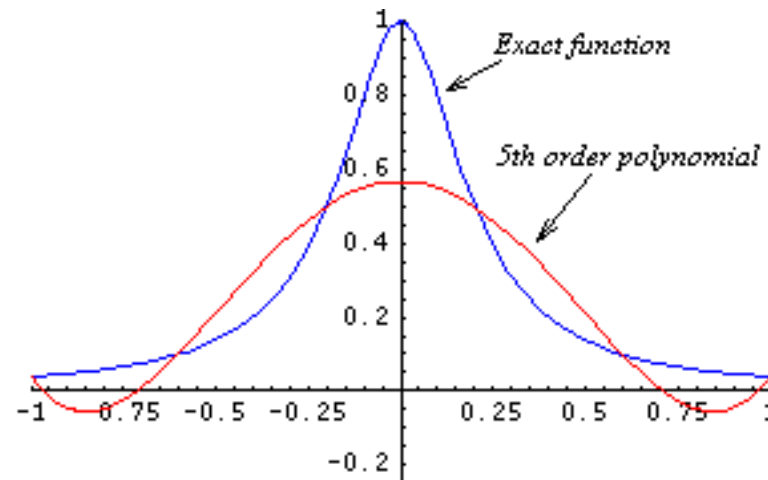


Figure : 5th order polynomial vs. exact function

Why Splines ?

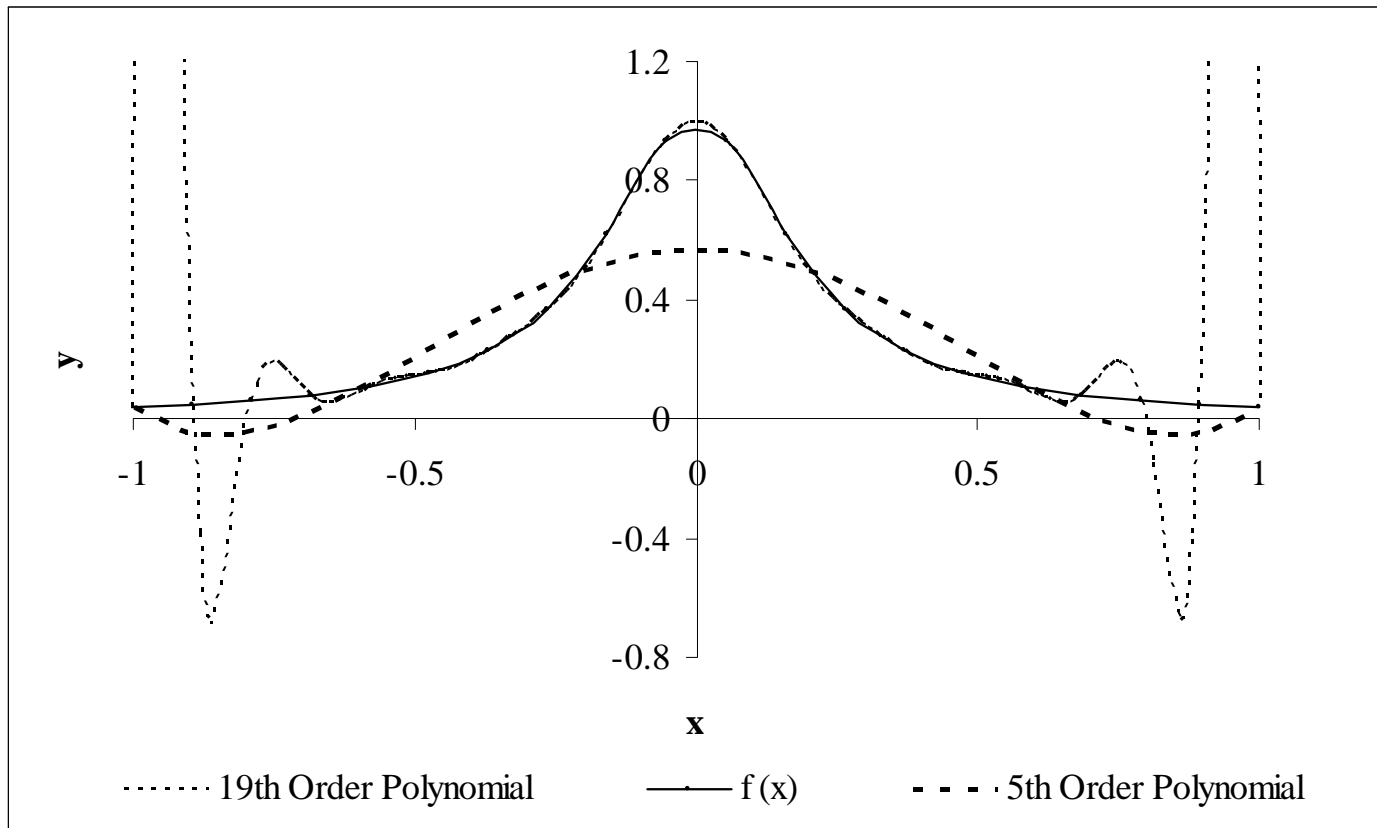


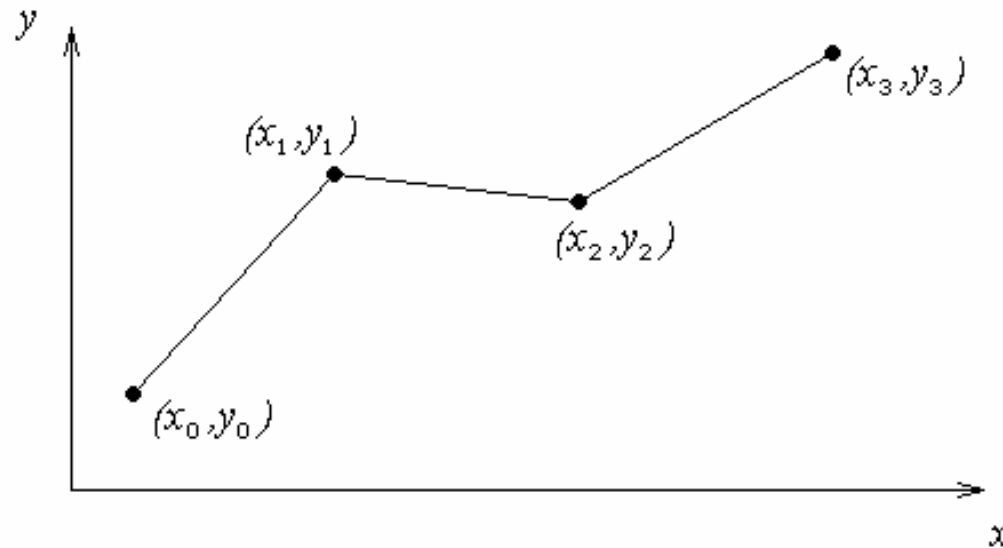
Figure : Higher order polynomial interpolation is a bad idea



Linear Interpolation

Given $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $(y_i = f(x_i))$

Figure : Linear splines





Linear Interpolation (contd)

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \quad x_0 \leq x \leq x_1$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \quad x_1 \leq x \leq x_2$$

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$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \leq x \leq x_n$$

Note the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between x_{i-1} and x_i .

Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at $t=16$ seconds using linear splines.

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Table : Velocity as a function of time

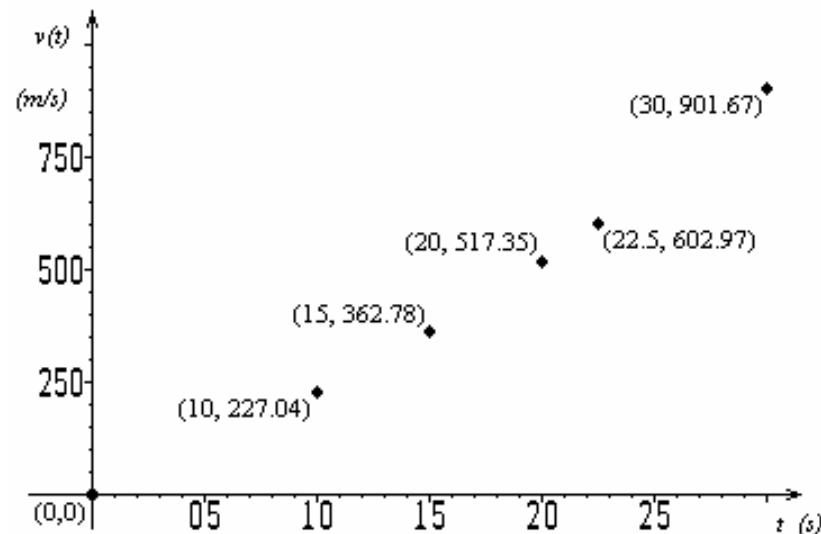


Figure : Velocity vs. time data for the rocket example



Linear Interpolation

$$t_0 = 15, \quad v(t_0) = 362.78$$

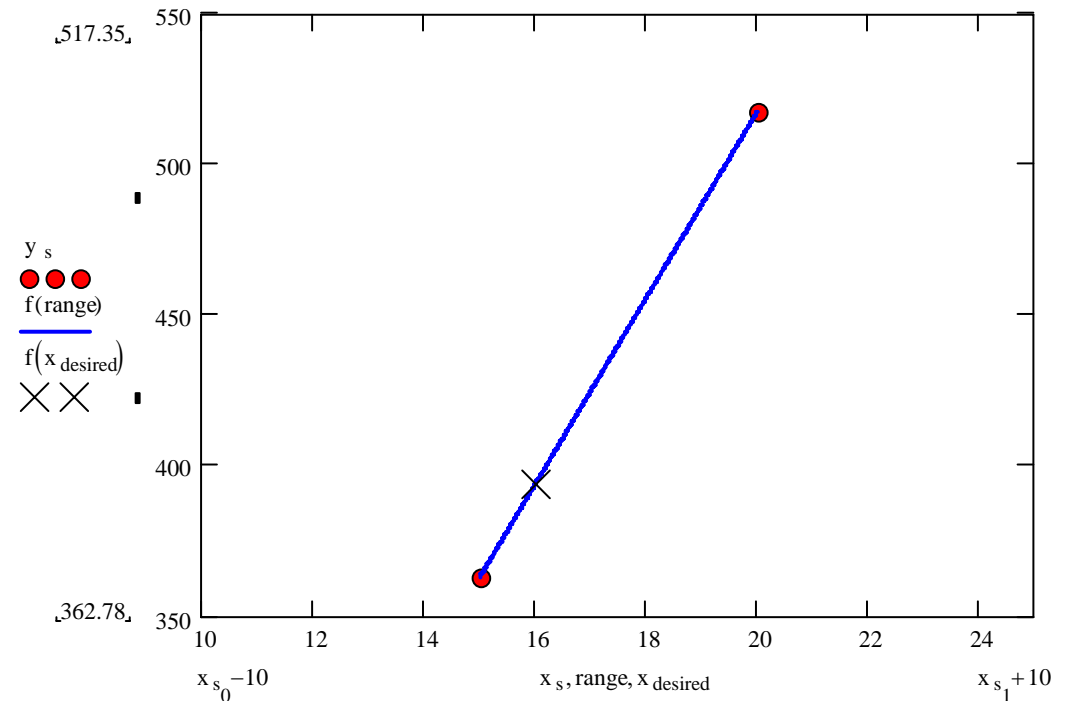
$$t_1 = 20, \quad v(t_1) = 517.35$$

$$\begin{aligned} v(t) &= v(t_0) + \frac{v(t_1) - v(t_0)}{t_1 - t_0} (t - t_0) \\ &= 362.78 + \frac{517.35 - 362.78}{20 - 15} (t - 15) \end{aligned}$$

$$v(t) = 362.78 + 30.913(t - 15)$$

At $t = 16$,

$$\begin{aligned} v(16) &= 362.78 + 30.913(16 - 15) \\ &= 393.7 \text{ m/s} \end{aligned}$$



Quadratic Interpolation

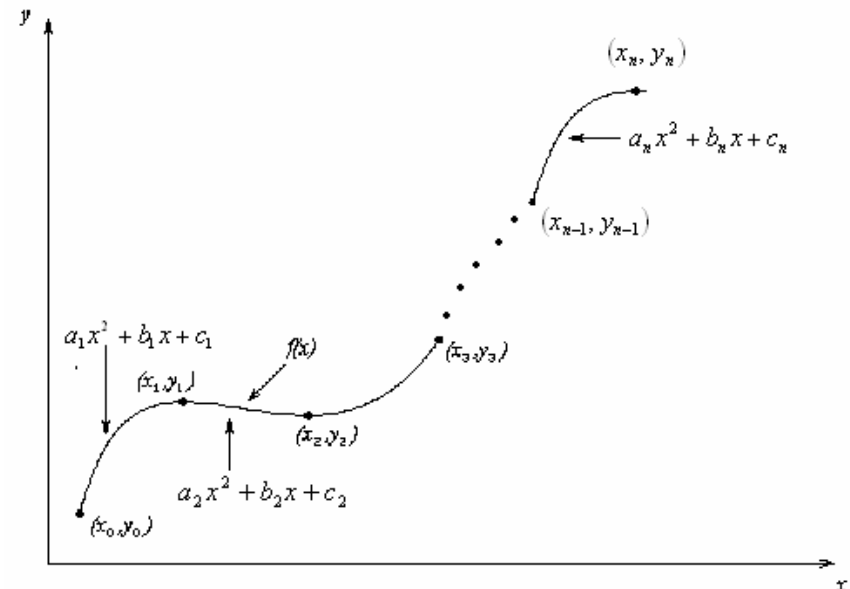
Given $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by

$$f(x) = a_1x^2 + b_1x + c_1, \quad x_0 \leq x \leq x_1$$

$$= a_2x^2 + b_2x + c_2, \quad x_1 \leq x \leq x_2$$

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$$= a_nx^2 + b_nx + c_n, \quad x_{n-1} \leq x \leq x_n$$



Find $a_i, b_i, c_i, i = 1, 2, \dots, n$

Quadratic Interpolation (contd)

Each quadratic spline goes through two consecutive data points

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$

$$a_1 x_1^2 + b_1 x_1 + c_1 = f(x_1)$$

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$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

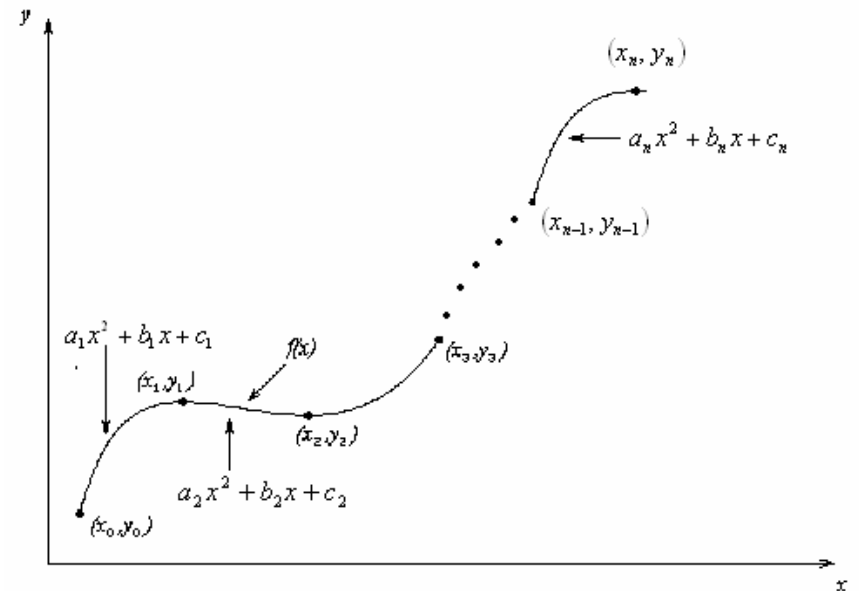
$$a_i x_i^2 + b_i x_i + c_i = f(x_i)$$

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$$a_n x_{n-1}^2 + b_n x_{n-1} + c_n = f(x_{n-1})$$

$$a_n x_n^2 + b_n x_n + c_n = f(x_n)$$



This condition gives $2n$ equations

Quadratic Splines (contd)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1 \text{ is } 2a_1x + b_1$$

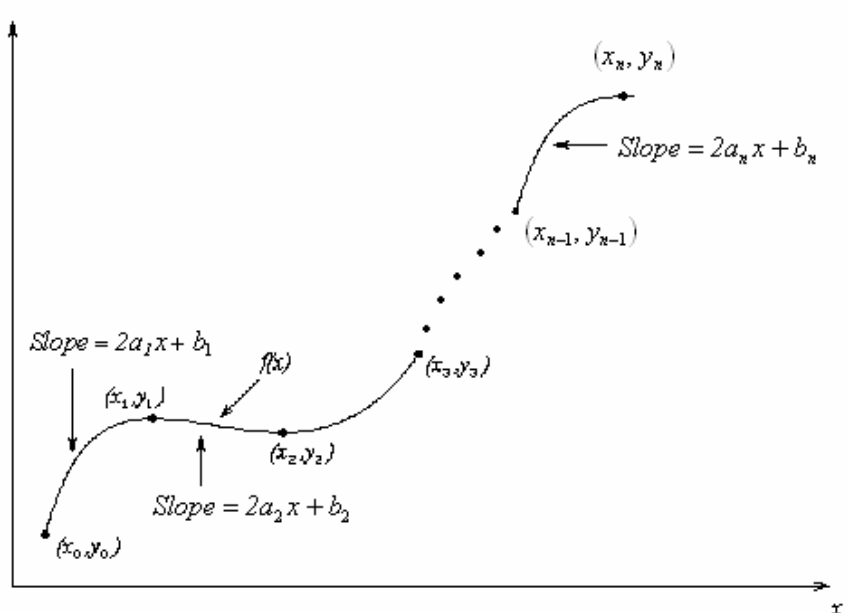
The derivative of the second spline

$$a_2x^2 + b_2x + c_2 \text{ is } 2a_2x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



Quadratic Splines (contd)

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

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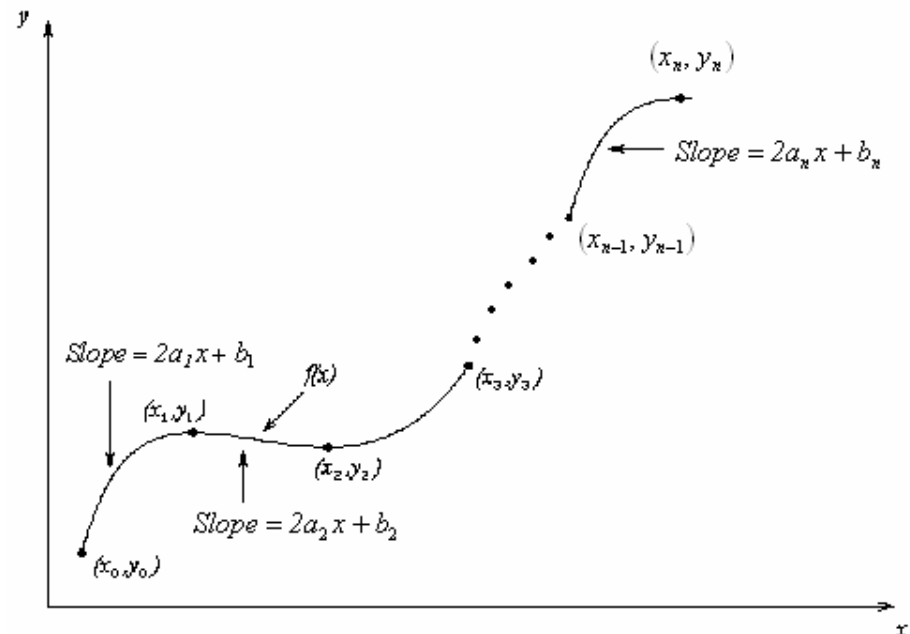
$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

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$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$



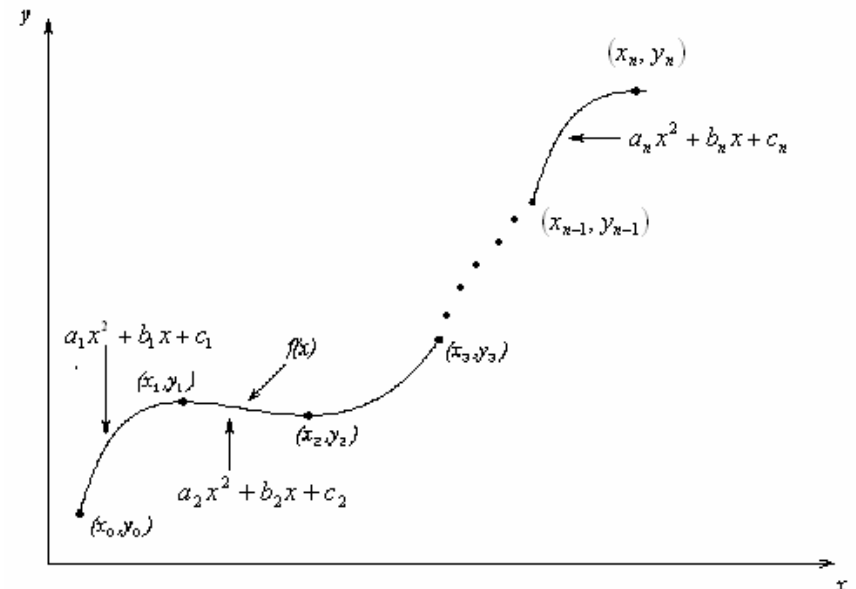
We have $(n-1)$ such equations. The total number of equations is $(2n) + (n-1) = (3n-1)$.

We can assume that the first spline is linear, that is $a_1 = 0$

Quadratic Splines (contd)

This gives us '3n' equations and '3n' unknowns. Once we find the '3n' constants, we can find the function at any value of 'x' using the splines,

$$\begin{aligned}
 f(x) &= a_1x^2 + b_1x + c_1, & x_0 \leq x \leq x_1 \\
 &= a_2x^2 + b_2x + c_2, & x_1 \leq x \leq x_2 \\
 &\cdot \\
 &\cdot \\
 &\cdot \\
 &= a_nx^2 + b_nx + c_n, & x_{n-1} \leq x \leq x_n
 \end{aligned}$$



Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at $t=16$ seconds using quadratic splines.

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Table : Velocity as a function of time

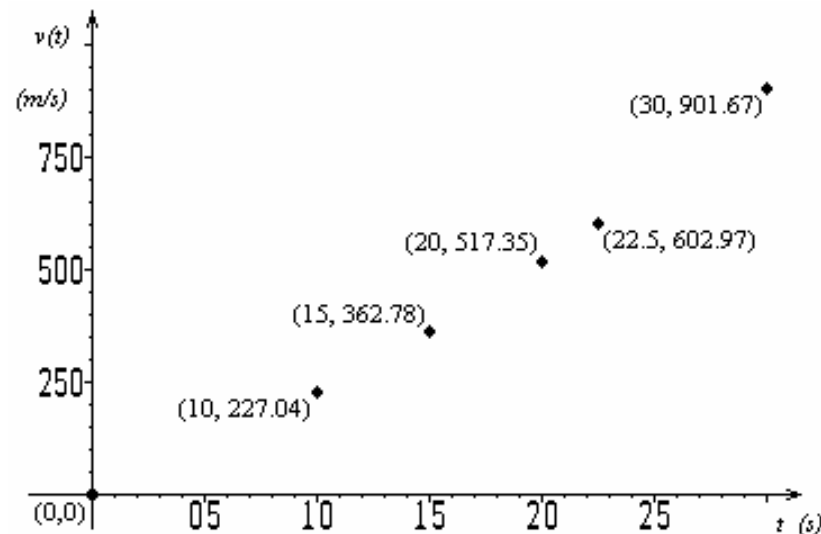


Figure : Velocity vs. time data for the rocket example



Solution

Since there are six data points, five quadratic splines pass through them.

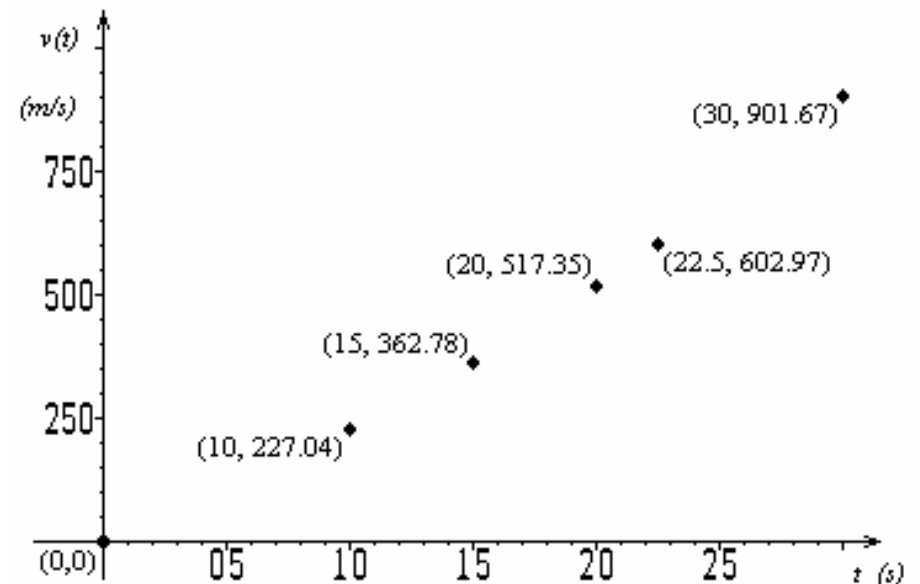
$$v(t) = a_1 t^2 + b_1 t + c_1, \quad 0 \leq t \leq 10$$

$$= a_2 t^2 + b_2 t + c_2, \quad 10 \leq t \leq 15$$

$$= a_3 t^2 + b_3 t + c_3, \quad 15 \leq t \leq 20$$

$$= a_4 t^2 + b_4 t + c_4, \quad 20 \leq t \leq 22.5$$

$$= a_5 t^2 + b_5 t + c_5, \quad 22.5 \leq t \leq 30$$





Solution (contd)

Setting up the equations

1. Each quadratic spline passes through two consecutive data points giving

$a_1 t^2 + b_1 t + c_1$ passes through $t = 0$ and $t = 10$,

$$a_1(0)^2 + b_1(0) + c_1 = 0 \quad (1)$$

$$a_1(10)^2 + b_1(10) + c_1 = 227.04 \quad (2)$$

Similarly,

$$a_2(10)^2 + b_2(10) + c_2 = 227.04 \quad (3)$$

$$a_2(15)^2 + b_2(15) + c_2 = 362.78 \quad (4)$$

$$a_3(15)^2 + b_3(15) + c_3 = 362.78 \quad (5)$$

$$a_3(20)^2 + b_3(20) + c_3 = 517.35 \quad (6)$$

$$a_4(20)^2 + b_4(20) + c_4 = 517.35 \quad (7)$$

$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97 \quad (8)$$

$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97 \quad (9)$$

$$a_5(30)^2 + b_5(30) + c_5 = 901.67 \quad (10)$$

Solution (contd)

Quadratic splines have continuous derivatives at the interior data points

At $t = 10$

$$2a_1(10) + b_1 - 2a_2(10) - b_2 = 0 \quad (11)$$

At $t = 15$

$$2a_2(15) + b_2 - 2a_3(15) - b_3 = 0 \quad (12)$$

At $t = 20$

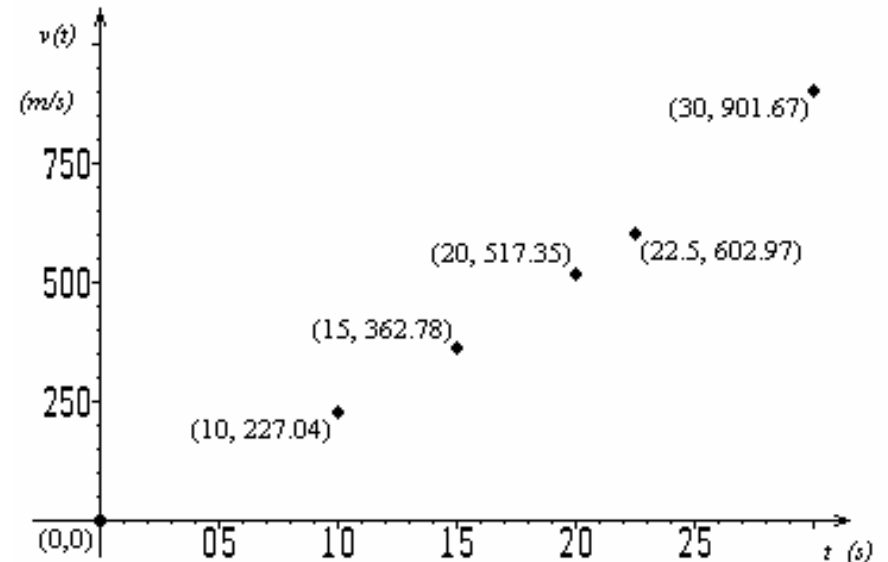
$$2a_3(20) + b_3 - 2a_4(20) - b_4 = 0 \quad (13)$$

At $t = 22.5$

$$2a_4(22.5) + b_4 - 2a_5(22.5) - b_5 = 0 \quad (14)$$

And assuming the first spline $a_1 t^2 + b_1 t + c_1$ is linear,

$$a_1 = 0 \quad (15)$$



Solution (contd)

$$\begin{bmatrix}
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 & 0 \\
 20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}
 \begin{bmatrix}
 a_1 \\
 b_1 \\
 c_1 \\
 a_2 \\
 b_2 \\
 c_2 \\
 a_3 \\
 b_3 \\
 c_3 \\
 a_4 \\
 b_4 \\
 c_4 \\
 a_5 \\
 b_5 \\
 c_5
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 227.04 \\
 227.04 \\
 362.78 \\
 362.78 \\
 517.35 \\
 517.35 \\
 602.97 \\
 602.97 \\
 901.67 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{bmatrix}$$



Solution (contd)

Solving the above 15 equations gives the 15 unknowns as

i	a_i	b_i	c_i
1	0	22.704	0
2	0.8888	4.928	88.88
3	-0.1356	35.66	-141.61
4	1.6048	-33.956	554.55
5	0.20889	28.86	-152.13

Solution (contd)

Therefore, the splines are given by

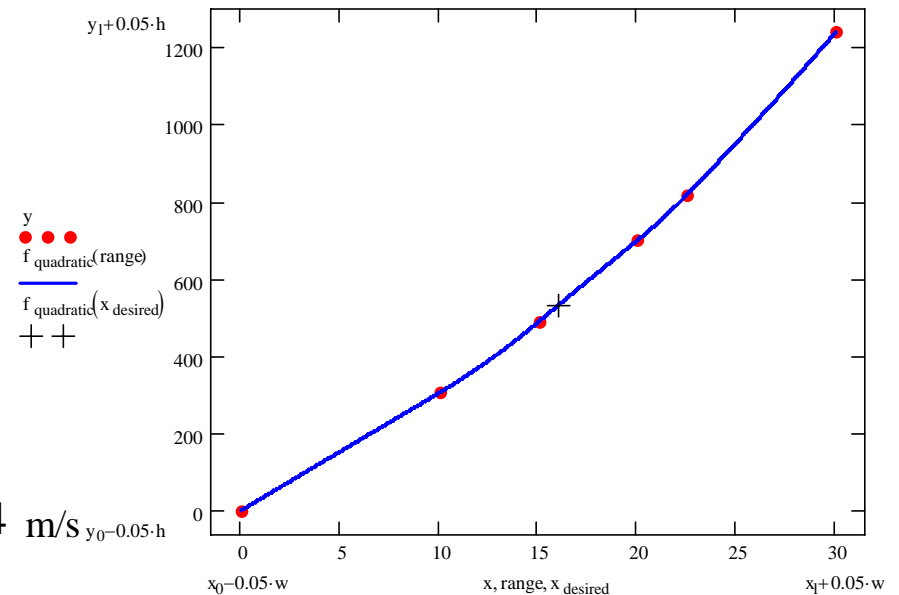
$$\begin{aligned}
 v(t) &= 22.704t, & 0 \leq t \leq 10 \\
 &= 0.8888t^2 + 4.928t + 88.88, & 10 \leq t \leq 15 \\
 &= -0.1356t^2 + 35.66t - 141.61, & 15 \leq t \leq 20 \\
 &= 1.6048t^2 - 33.956t + 554.55, & 20 \leq t \leq 22.5 \\
 &= 0.20889t^2 + 28.86t - 152.13, & 22.5 \leq t \leq 30
 \end{aligned}$$

At $t = 16$

$$v(16) = -0.1356(16)^2 + 35.66(16) - 141.61 = 394.24 \text{ m/s}$$

The absolute relative approximate error, $|\epsilon_a|$ is

$$\begin{aligned}
 |\epsilon_a| &= \left| \frac{394.24 - 393.7}{394.24} \right| \times 100 \\
 &= 0.1369\%
 \end{aligned}$$





Distance from Velocity Profile

Find the distance covered by the rocket from $t=11$ s to $t=16$ s

$$s(16) - s(11) = \int_{11}^{16} v(t) dt$$

But since the splines are valid over different ranges, we need to break the integral accordingly as

$$v(t) = 0.8888t^2 + 4.928t + 88.88, \quad 10 \leq t \leq 15$$

$$= -0.1356t^2 + 35.66t - 141.61, \quad 15 \leq t \leq 20$$

$$\int_{11}^{16} v(t) dt = \int_{11}^{15} v(t) dt + \int_{15}^{16} v(t) dt$$

$$s(16) - s(11) = \int_{11}^{15} (0.8888t^2 + 4.928t + 88.88) dt + \int_{15}^{16} (-0.1356t^2 + 35.66t - 141.61) dt$$

$$= 1595.9 \text{ m}$$



Acceleration from Velocity Profile

Find the acceleration of the rocket at $t=16$ s.

$$a(16) = \left. \frac{d}{dt} v(t) \right|_{t=16}$$

The quadratic spline valid at $t=16$ is given by

$$v(t) = -0.1356t^2 + 35.66t - 141.61, \quad 15 \leq t \leq 20$$

$$a(t) = \frac{d}{dt} v(t) = \frac{d}{dt} (-0.1356t^2 + 35.66t - 141.61)$$

$$a(t) = -0.2712t + 35.66 \quad 15 \leq t \leq 20$$

$$\begin{aligned} a(16) &= -0.2712(16) + 35.66 \\ &= 31.321 \text{ m/s}^2 \end{aligned}$$