Chapter 07.05
Gauss Quadrature Rule of Integration

After reading this chapter, you should be able to:

1. *derive the Gauss quadrature method for integration and be able to use it to solve problems, and*
2. *use Gauss quadrature method to solve examples of approximate integrals.*

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral.

Here, we will discuss the Gauss quadrature rule of approximating integrals of the form

$$I = \int_a^b f(x)dx$$

where

$f(x)$ is called the integrand,
$a =$ lower limit of integration
$b =$ upper limit of integration

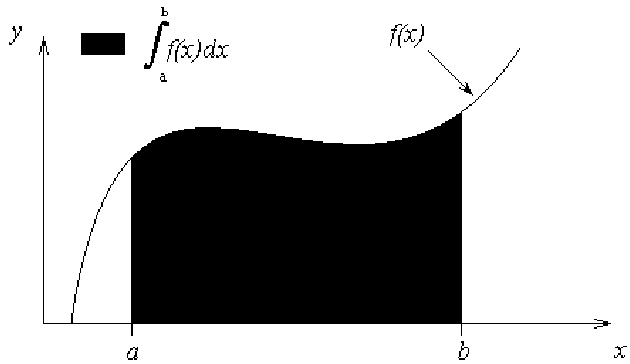

Figure 1 Integration of a function.

Gauss Quadrature Rule

Background:
To derive the trapezoidal rule from the method of undetermined coefficients, we approximated

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b) \tag{1}$$

Let the right hand side be exact for integrals of a straight line, that is, for an integrated form of

$$\int_a^b (a_0 + a_1 x)dx$$

So

$$\int_a^b (a_0 + a_1 x)dx = \left[a_0 x + a_1 \frac{x^2}{2}\right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) \tag{2}$$

But from Equation (1), we want

$$\int_a^b (a_0 + a_1 x)dx = c_1 f(a) + c_2 f(b) \tag{3}$$

to give the same result as Equation (2) for $f(x) = a_0 + a_1 x$.

$$\int_a^b (a_0 + a_1 x)dx = c_1(a_0 + a_1 a) + c_2(a_0 + a_1 b)$$

$$= a_0(c_1 + c_2) + a_1(c_1 a + c_2 b) \tag{4}$$

Hence from Equations (2) and (4),

$$a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) = a_0(c_1 + c_2) + a_1(c_1 a + c_2 b)$$

Since a_0 and a_1 are arbitrary constants for a general straight line

$$c_1 + c_2 = b - a \tag{5a}$$

$$c_1 a + c_2 b = \frac{b^2 - a^2}{2} \tag{5b}$$

Multiplying Equation (5a) by a and subtracting from Equation (5b) gives

$$c_2 = \frac{b-a}{2} \tag{6a}$$

Substituting the above found value of c_2 in Equation (5a) gives

$$c_1 = \frac{b-a}{2} \tag{6b}$$

Therefore

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b) \tag{7}$$

Derivation of two-point Gauss quadrature rule

<u>Method 1:</u>

The two-point Gauss quadrature rule is an extension of the trapezoidal rule approximation where the arguments of the function are not predetermined as a and b, but as unknowns x_1 and x_2. So in the two-point Gauss quadrature rule, the integral is approximated as

$$I = \int_a^b f(x)dx$$

$$\approx c_1 f(x_1) + c_2 f(x_2)$$

There are four unknowns x_1, x_2, c_1 and c_2. These are found by assuming that the formula gives exact results for integrating a general third order polynomial, $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. Hence

$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4}\right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right) \tag{8}$$

The formula would then give

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) =$$

$$c_1\left(a_0 + a_1 x_1 + a_2 x_2^2 + a_3 x_3^3\right) + c_2\left(a_0 + a_1 x_1 + a_2 x_2^2 + a_3 x_3^3\right) \tag{9}$$

Equating Equations (8) and (9) gives

$$a_0(b-a)+a_1\left(\frac{b^2-a^2}{2}\right)+a_2\left(\frac{b^3-a^3}{3}\right)+a_3\left(\frac{b^4-a^4}{4}\right)$$
$$=c_1\left(a_0+a_1x_1+a_2x_1^{\;2}+a_3x_1^{\;3}\right)+c_2\left(a_0+a_1x_2+a_2x_2^{\;2}+a_3x_2^{\;3}\right)$$
$$=a_0(c_1+c_2)+a_1(c_1x_1+c_2x_2)+a_2\left(c_1x_1^{\;2}+c_2x_2^{\;2}\right)+a_3\left(c_1x_1^{\;3}+c_2x_2^{\;3}\right) \quad (10)$$

Since in Equation (10), the constants a_0, a_1, a_2, and a_3 are arbitrary, the coefficients of a_0, a_1, a_2, and a_3 are equal. This gives us four equations as follows.

$$b-a=c_1+c_2$$
$$\frac{b^2-a^2}{2}=c_1x_1+c_2x_2$$
$$\frac{b^3-a^3}{3}=c_1x_1^{\;2}+c_2x_2^{\;2}$$
$$\frac{b^4-a^4}{4}=c_1x_1^{\;3}+c_2x_2^{\;3} \quad (11)$$

Without proof (see Example 1 for proof of a related problem), we can find that the above four simultaneous nonlinear equations have only one acceptable solution

$$c_1=\frac{b-a}{2}$$
$$c_2=\frac{b-a}{2}$$
$$x_1=\left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}$$
$$x_2=\left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2} \quad (12)$$

Hence

$$\int_a^b f(x)dx \approx c_1 f(x_1)+c_2 f(x_2)$$
$$=\frac{b-a}{2}f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\right)+\frac{b-a}{2}f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\right) \quad (13)$$

Method 2:

We can derive the same formula by assuming that the expression gives exact values for the individual integrals of $\int_a^b 1dx$, $\int_a^b xdx$, $\int_a^b x^2dx$, and $\int_a^b x^3dx$. The reason the formula can also be

derived using this method is that the linear combination of the above integrands is a general third order polynomial given by $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.
These will give four equations as follows

$$\int_a^b 1dx = b - a = c_1 + c_2$$

$$\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1x_1 + c_2x_2$$

$$\int_a^b x^2dx = \frac{b^3 - a^3}{3} = c_1x_1^{\ 2} + c_2x_2^{\ 2}$$

$$\int_a^b x^3dx = \frac{b^4 - a^4}{4} = c_1x_1^{\ 3} + c_2x_2^{\ 3} \quad (14)$$

These four simultaneous nonlinear equations can be solved to give a single acceptable solution

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2}$$

$$x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2} \quad (15)$$

Hence

$$\int_a^b f(x)dx \approx \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) \quad (16)$$

Since two points are chosen, it is called the two-point Gauss quadrature rule. Higher point versions can also be developed.

Higher point Gauss quadrature formulas

For example

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) \quad (17)$$

is called the three-point Gauss quadrature rule. The coefficients c_1, c_2 and c_3, and the function arguments x_1, x_2 and x_3 are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial

$$\int_a^b \left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5\right)dx.$$

General n-point rules would approximate the integral

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots\ldots + c_n f(x_n) \tag{18}$$

Arguments and weighing factors for *n*-point Gauss quadrature rules

In handbooks (see Table 1), coefficients and arguments given for n-point Gauss quadrature rule are given for integrals of the form

$$\int_{-1}^{1} g(x)dx \approx \sum_{i=1}^{n} c_i g(x_i) \tag{19}$$

Table 1 Weighting factors c and function arguments x used in Gauss quadrature formulas

Points	Weighting Factors	Function Arguments
2	$c_1 = 1.000000000$	$x_1 = -0.577350269$
	$c_2 = 1.000000000$	$x_2 = 0.577350269$
3	$c_1 = 0.555555556$	$x_1 = -0.774596669$
	$c_2 = 0.888888889$	$x_2 = 0.000000000$
	$c_3 = 0.555555556$	$x_3 = 0.774596669$
4	$c_1 = 0.347854845$	$x_1 = -0.861136312$
	$c_2 = 0.652145155$	$x_2 = -0.339981044$
	$c_3 = 0.652145155$	$x_3 = 0.339981044$
	$c_4 = 0.347854845$	$x_4 = 0.861136312$
5	$c_1 = 0.236926885$	$x_1 = -0.906179846$
	$c_2 = 0.478628670$	$x_2 = -0.538469310$
	$c_3 = 0.568888889$	$x_3 = 0.000000000$
	$c_4 = 0.478628670$	$x_4 = 0.538469310$
	$c_5 = 0.236926885$	$x_5 = 0.906179846$
6	$c_1 = 0.171324492$	$x_1 = -0.932469514$
	$c_2 = 0.360761573$	$x_2 = -0.661209386$
	$c_3 = 0.467913935$	$x_3 = -0.238619186$
	$c_4 = 0.467913935$	$x_4 = 0.238619186$

	$c_5 = 0.360761573$ $c_6 = 0.171324492$	$x_5 = 0.661209386$ $x_6 = 0.932469514$

So if the table is given for $\int_{-1}^{1} g(x)dx$ integrals, how does one solve $\int_{a}^{b} f(x)dx$?

The answer lies in that any integral with limits of $[a, b]$ can be converted into an integral with limits $[-1, 1]$. Let

$$x = mt + c \tag{20}$$

If $x = a,$ then $t = -1$

If $x = b,$ then $t = +1$

such that

$$a = m(-1) + c$$
$$b = m(1) + c \tag{21}$$

Solving the two Equations (21) simultaneously gives

$$m = \frac{b-a}{2}$$
$$c = \frac{b+a}{2} \tag{22}$$

Hence

$$x = \frac{b-a}{2}t + \frac{b+a}{2}$$
$$dx = \frac{b-a}{2}dt$$

Substituting our values of x and dx into the integral gives us

$$\int_{a}^{b} f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\frac{b-a}{2}dx \tag{23}$$

Example 1

For an integral $\int_{-1}^{1} f(x)dx,$ show that the two-point Gauss quadrature rule approximates to

$$\int_{-1}^{1} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

where

$$c_1 = 1$$
$$c_2 = 1$$
$$x_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = \frac{1}{\sqrt{3}}$$

Solution

Assuming the formula

$$\int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2) \quad \text{(E1.1)}$$

gives exact values for integrals $\int_{-1}^{1} 1dx,\ \int_{-1}^{1} xdx,\ \int_{-1}^{1} x^2 dx,$ and $\int_{-1}^{1} x^3 dx$. Then

$$\int_{-1}^{1} 1dx = 2 = c_1 + c_2 \quad \text{(E1.2)}$$

$$\int_{-1}^{1} xdx = 0 = c_1 x_1 + c_2 x_2 \quad \text{(E1.3)}$$

$$\int_{-1}^{1} x^2 dx = \frac{2}{3} = c_1 x_1^{\ 2} + c_2 x_2^{\ 2} \quad \text{(E1.4)}$$

$$\int_{-1}^{1} x^3 dx = 0 = c_1 x_1^{\ 3} + c_2 x_2^{\ 3} \quad \text{(E1.5)}$$

Multiplying Equation (E1.3) by $x_1^{\ 2}$ and subtracting from Equation (E1.5) gives

$$c_2 x_2 \left(x_1^{\ 2} - x_2^{\ 2}\right) = 0 \quad \text{(E1.6)}$$

The solution to the above equation is

$c_2 = 0,$ or/and

$x_2 = 0,$ or/and

$x_1 = x_2,$ or/and

$x_1 = -x_2.$

I. $c_2 = 0$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 = 2,\ c_1 x_1 = 0,$ $c_1 x_1^2 = \frac{2}{3},$ and $c_1 x_1^3 = 0$. But since $c_1 = 2$, then $x_1 = 0$ from $c_1 x_1 = 0$, but $x_1 = 0$ conflicts with $c_1 x_1^2 = \frac{2}{3}$.

II. $x_2 = 0$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 + c_2 = 2$, $c_1 x_1 = 0,$ $c_1 x_1^2 = \frac{2}{3},$ and $c_1 x_1^3 = 0$. Since $c_1 x_1 = 0$, then c_1 or x_1 has to be zero but this violates $c_1 x_1^2 = \frac{2}{3} \neq 0$.

III. $x_1 = x_2$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 + c_2 = 2$, $c_1 x_1 + c_2 x_1 = 0,$ $c_1 x_1^2 + c_2 x_1^{\ 2} = \frac{2}{3},$ and $c_1 x_1^3 + c_2 x_1^3 = 0$. If $x_1 \neq 0$, then $c_1 x_1 + c_2 x_1 = 0$

gives $c_1 + c_2 = 0$ and that violates $c_1 + c_2 = 2$. If $x_1 = 0$, then that violates $c_1 x_1^2 + c_2 x_1^2 = \frac{2}{3} \neq 0$.

That leaves the solution of $x_1 = -x_2$ as the only possible acceptable solution and in fact, it does not have violations (see it for yourself)

$$x_1 = -x_2 \tag{E1.7}$$

Substituting (E1.7) in Equation (E1.3) gives

$$c_1 = c_2 \tag{E1.8}$$

From Equations (E1.2) and (E1.8),

$$c_1 = c_2 = 1 \tag{E1.9}$$

Equations (E1.4) and (E1.9) gives

$$x_1^2 + x_2^2 = \frac{2}{3} \tag{E1.10}$$

Since Equation (E1.7) requires that the two results be of opposite sign, we get

$$x_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = \frac{1}{\sqrt{3}}$$

Hence

$$\int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2) \tag{E1.11}$$

$$= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Example 2

For an integral $\int_a^b f(x)dx$, derive the one-point Gauss quadrature rule.

Solution

The one-point Gauss quadrature rule is

$$\int_a^b f(x)dx \approx c_1 f(x_1) \tag{E2.1}$$

Assuming the formula gives exact values for integrals $\int_{-1}^{1} 1dx$, and $\int_{-1}^{1} xdx$

$$\int_a^b 1dx = b - a = c_1$$

$$\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1 x_1 \tag{E2.2}$$

Since $c_1 = b - a,$ the other equation becomes

$$(b-a)x_1 = \frac{b^2 - a^2}{2}$$

$$x_1 = \frac{b+a}{2} \quad \text{(E2.3)}$$

Therefore, one-point Gauss quadrature rule can be expressed as

$$\int_a^b f(x)dx \approx (b-a)f\left(\frac{b+a}{2}\right) \quad \text{(E2.4)}$$

Example 3

What would be the formula for

$$\int_a^b f(x)dx = c_1 f(a) + c_2 f(b)$$

if you want the above formula to give you exact values of $\int_a^b \left(a_0 x + b_0 x^2\right)dx,$ that is, a linear combination of x and x^2.

Solution

If the formula is exact for a linear combination of x and x^2, then

$$\int_a^b x dx = \frac{b^2 - a^2}{2} = c_1 a + c_2 b$$

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} = c_1 a^2 + c_2 b^2 \quad \text{(E3.1)}$$

Solving the two Equations (E3.1) simultaneously gives

$$\begin{bmatrix} a & b \\ a^2 & b^2 \end{bmatrix} \begin{bmatrix} \mathrm{c}_1 \\ \mathrm{c}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{b^2 - a^2}{2} \\ \dfrac{b^3 - a^3}{3} \end{bmatrix}$$

$$c_1 = -\frac{1}{6}\frac{-ab - b^2 + 2a^2}{a}$$

$$c_2 = -\frac{1}{6}\frac{a^2 + ab - 2b^2}{b} \quad \text{(E3.2)}$$

So

$$\int_a^b f(x)dx = -\frac{1}{6}\frac{-ab - b^2 + 2a^2}{a} f(a) - \frac{1}{6}\frac{a^2 + ab - 2b^2}{b} f(b) \quad \text{(E3.3)}$$

Let us see if the formula works.

Evaluate $\int_2^5 \left(2x^2 - 3x\right)dx$ using Equation(E3.3)

$$\int_2^5 \left(2x^2 - 3x\right)dx \approx c_1 f(a) + c_2 f(b)$$
$$= -\frac{1}{6}\frac{-(2)(5)-5^2+2(2)^2}{2}\left[2(2)^2-3(2)\right] - \frac{1}{6}\frac{2^2+2(5)-2(5)^2}{5}\left[2(5)^2-3(5)\right]$$
$$= 46.5$$

The exact value of $\int_2^5 \left(2x^2 - 3x\right)dx$ is given by

$$\int_2^5 \left(2x^2 - 3x\right)dx = \left[\frac{2x^3}{3} - \frac{3x^2}{2}\right]_2^5$$
$$= 46.5$$

Any surprises?

Now evaluate $\int_2^5 3dx$ using Equation (E3.3)

$$\int_2^5 3dx \approx c_1 f(a) + c_2 f(b)$$
$$= -\frac{1}{6}\frac{-2(5)-5^2+2(2)^2}{2}(3) - \frac{1}{6}\frac{2^2+2(5)-2(5)^2}{5}(3)$$
$$= 10.35$$

The exact value of $\int_2^5 3dx$ is given by

$$\int_2^5 3dx = \left[3x\right]_2^5$$
$$= 9$$

Because the formula will only give exact values for linear combinations of x and x^2, it does not work exactly even for a simple integral of $\int_2^5 3dx$.

Do you see now why we choose $a_0 + a_1 x$ as the integrand for which the formula

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

gives us exact values?

Example 4

Use two-point Gauss quadrature rule to approximate the distance covered by a rocket from $t = 8$ to $t = 30$ as given by

$$x = \int_8^{30} \left(2000 \ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t\right)dt$$

Also, find the absolute relative true error.

Solution

First, change the limits of integration from $[8, 30]$ to $[-1, 1]$ using Equation(23) gives

$$\int_8^{30} f(t)dt = \frac{30-8}{2}\int_{-1}^{1} f\left(\frac{30-8}{2}x + \frac{30+8}{2}\right)dx$$

$$= 11\int_{-1}^{1} f(11x+19)dx$$

Next, get weighting factors and function argument values from Table 1 for the two point rule,

$c_1 = \ \ 1.000000000.$

$x_1 = -0.577350269$

$c_2 = \ \ 1.000000000$

$x_2 = \ \ 0.577350269$

Now we can use the Gauss quadrature formula

$$11\int_{-1}^{1} f(11x+19)dx \approx 11[c_1 f(11x_1+19) + c_2 f(11x_2+19)]$$

$$= 11[f(11(-0.5773503)+19) + f(11(0.5773503)+19)]$$

$$= 11[f(12.64915) + f(25.35085)]$$

$$= 11[(296.8317) + (708.4811)]$$

$$= 11058.44 \text{ m}$$

since

$$f(12.64915) = 2000\ln\left[\frac{140000}{140000-2100(12.64915)}\right] - 9.8(12.64915)$$

$$= 296.8317$$

$$f(25.35085) = 2000\ln\left[\frac{140000}{140000-2100(25.35085)}\right] - 9.8(25.35085)$$

$$= 708.4811$$

The absolute relative true error, $|\in_t|$, is (True value = 11061.34 m)

$$|\in_t| = \left|\frac{11061.34 - 11058.44}{11061.34}\right| \times 100$$

$$= 0.0262\%$$

Example 5

Use three-point Gauss quadrature rule to approximate the distance covered by a rocket from $t = 8$ to $t = 30$ as given by

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t\right)dt$$

Also, find the absolute relative true error.

Solution

First, change the limits of integration from $[8, 30]$ to $[-1, 1]$ using Equation (23) gives

$$\int_{8}^{30} f(t)dt = \frac{30-8}{2}\int_{-1}^{1} f\left(\frac{30-8}{2}x + \frac{30+8}{2}\right)dx$$
$$= 11\int_{-1}^{1} f(11x+19)dx$$

The weighting factors and function argument values are

$$c_1 = 0.555555556$$
$$x_1 = -0.774596669$$
$$c_2 = 0.888888889$$
$$x_2 = 0.000000000$$
$$c_3 = 0.555555556$$
$$x_3 = 0.774596669$$

and the formula is

$$11\int_{-1}^{1} f(11x+19)dx \approx 11\left[c_1 f(11x_1+19) + c_2 f(11x_2+19) + c_3 f(11x_3+19)\right]$$
$$= 11\begin{bmatrix} 0.5555556 f(11(-.7745967)+19) + 0.8888889 f(11(0.0000000)+19) \\ + 0.5555556 f(11(0.7745967)+19) \end{bmatrix}$$
$$= 11[0.55556 f(10.47944) + 0.88889 f(19.00000) + 0.55556 f(27.52056)]$$
$$= 11[0.55556 \times 239.3327 + 0.88889 \times 484.7455 + 0.55556 \times 795.1069]$$
$$= 11061.31 \text{ m}$$

since

$$f(10.47944) = 2000\ln\left[\frac{140000}{140000 - 2100(10.47944)}\right] - 9.8(10.47944)$$
$$= 239.3327$$
$$f(19.00000) = 2000\ln\left[\frac{140000}{140000 - 2100(19.00000)}\right] - 9.8(19.00000)$$
$$= 484.7455$$
$$f(27.52056) = 2000\ln\left[\frac{140000}{140000 - 2100(27.52056)}\right] - 9.8(27.52056)$$
$$= 795.1069$$

The absolute relative true error, $|\in_t|$, is (True value = 11061.34 m)

$$|\in_t| = \left|\frac{11061.34 - 11061.31}{11061.34}\right| \times 100$$
$$= 0.0003\%$$

INTEGRATION	
Topic	Gauss quadrature rule
Summary	These are textbook notes of Gauss quadrature rule
Major	General Engineering
Authors	Autar Kaw, Michael Keteltas
Date	December 7, 2008
Web Site	http://numericalmethods.eng.usf.edu