Holistic Numerical Methods Institute

Ordinary Differential Equations

Topic: Finite Difference Method

Major: General Engineering

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Finite Difference Method

An example of a boundary value ordinary differential equation is

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{d\ u}{dr} - \frac{u}{r^2} = 0,\ u(5) = 0.008731'',\ u(8) = 0.0030769''$$

The derivatives in such ordinary differential equation are substituted by finite divided differences approximations, such as

$$\frac{dy}{dx} \approx \frac{y_{i+1} - y_i}{\Delta x}$$

$$\frac{d^2y}{dx^2} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}$$

Example

Take the case of a pressure vessel that is being tested in the laboratory to check its ability to withstand pressure. For a thick pressure vessel of inner radius *a* and outer radius *b*, the differential equation for the radial displacement *u* of a point along the thickness is given by

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=0$$

The pressure vessel can be modeled as,

$$\frac{d^2u}{dr^2} \approx \frac{u_{i+1}-2u_i+u_{i-1}}{(\Delta r)^2}$$

$$\frac{du}{dr} \approx \frac{u_{i+1}-u_i}{\Delta r}$$

Substituting these approximations gives you,

$$\frac{u_{i+1}-2u_i+u_{i-1}}{(\Delta r)^2}+\frac{1}{r_i}\frac{u_{i+1}-u_i}{\Delta r}-\frac{u_i}{r_i^2}=0$$

$$\left(\frac{1}{(\Delta r)^2}+\frac{1}{r_i\Delta r}\right)u_{i+1}+\left(-\frac{2}{(\Delta r)^2}-\frac{1}{r_i\Delta r}-\frac{1}{r_i^2}\right)u_i+\frac{1}{(\Delta r)^2}u_{i-1}=0$$

Solution

Step 1 At node $i = 0, \quad r_0 = a = 5'' \quad u_0 = 0.0038731''$

Step 2 At node $i = 1, \quad r_1 = r_0 + \Delta r = 5 + 0.6 = 5.6''$

$$\frac{1}{(0.6)^2}u_0 + \left(-\frac{2}{(0.6)^2} - \frac{1}{(5.6)(0.6)} - \frac{1}{(5.6)^2}\right)u_1 + \left(\frac{1}{0.6^2} + \frac{1}{(5.6)(0.6)}\right)u_2 = 0$$

$$2.7778u_0 - 5.8851u_1 + 3.0754u_2 = 0$$

Step 3 At node $i = 2, \quad r_2 = r_1 + \Delta r = 5.6 + 0.6 = 6.2''$

$$\frac{1}{0.6^2}u_1 + \left(-\frac{2}{0.6^2} - \frac{1}{(6.2)(0.6)} - \frac{1}{6.2^2}\right)u_2 + \left(\frac{1}{0.6^2} + \frac{1}{(6.2)(0.6)}\right)u_3 = 0$$

$$2.7778u_1 - 5.8504u_2 + 3.0466u_3 = 0$$

Solution Cont

Step 4 At node $i = 3, \quad r_3 = r_2 + \Delta r = 6.2 + 0.6 = 6.8"$

$$\frac{1}{0.6^2}u_2 + \left(-\frac{2}{0.6^2} - \frac{1}{(6.8)(0.6)} - \frac{1}{6.8^2}\right)u_3 + \left(\frac{1}{0.6^2} + \frac{1}{(6.8)(0.6)}\right)u_4 = 0$$

$$2.7778u_2 - 5.8223u_3 + 3.0229u_4 = 0$$

Step 5 At node $i = 4, \quad r_4 = r_3 + \Delta r = 6.8 + 0.6 = 7.4"$

$$\frac{1}{0.6^2}u_3 + \left(-\frac{2}{0.6^2} - \frac{1}{(7.4)(0.6)} - \frac{1}{(7.4)^2}\right)u_4 + \left(\frac{1}{0.6^2} + \frac{1}{(7.4)(0.6)}\right)u_5 = 0$$

$$2.7778u_3 - 5.7990u_4 + 3.0030u_5 = 0$$

Step 6 At node $i = 5, \quad r_5 = r_4 + \Delta r = 7.4 + 0.6 = 8$

$$u_5 = u/_{r=b} = 0.0030769$$

Solving system of equations

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2.7778 & -5.8851 & 3.0754 & 0 & 0 & 0 \\ 0 & 2.7778 & -5.8504 & 3.0466 & 0 & 0 \\ 0 & 0 & 2.7778 & -5.8223 & 3.0229 & 0 \\ 0 & 0 & 0 & 2.7778 & -5.7990 & 3.0030 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0.0038731 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.0030769 \end{bmatrix}$$

$u_0 = 0.0038731$

$u_1 = 0.0036165$

$u_2 = 0.0034222$

$u_3 = 0.0032743$

$u_4 = 0.0031618$

$u_5 = 0.0030769$

Solution Cont

$$\left.\frac{du}{dr}\right|_{r=a} \approx \frac{u_1 - u_0}{\Delta r} = \frac{0.0036165 - 0.0038731}{0.6} = -0.00042767$$

$$\sigma_{\max} = \frac{30 \times 10^6}{1 - 0.3^2}\left(\frac{0.0038731}{5} + 0.3(-0.00042767)\right) = 21307\ psi$$

$$FS = \frac{36 \times 10^3}{21307} = 1.6896$$

$$E_t = 20538 - 21307 = -768.59$$

$$|\in_t| = \left|\frac{20538 - 21307}{20538}\right| \times 100 = 3.7443\%$$

Solution Cont

Using the approximation of

$$\frac{d^2y}{dx^2} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} \quad \text{and} \quad \frac{dy}{dx} \approx \frac{y_{i+1} - y_{i-1}}{2(\Delta x)}$$

Gives you

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} + \frac{1}{r_i}\frac{u_{i+1} - u_{i-1}}{2(\Delta r)} - \frac{u_i}{r_i{}^2} = 0$$

$$\left(-\frac{1}{2r_i(\Delta r)} + \frac{1}{(\Delta r)^2}\right)u_{i-1} + \left(-\frac{2}{(\Delta r)^2} - \frac{1}{r_i{}^2}\right)u_i + \left(\frac{1}{(\Delta r)^2} + \frac{1}{2r_i\Delta r}\right)u_{i+1} = 0$$

Solution Cont

Step 1 At node $i = 0, r_0 = a = 5$

$$u_0 = 0.0038731$$

Step 2 At node $i = 1,\ r_1 = r_0 + \Delta r = 5 + 0.6 = 5.6''$

$$\left(-\frac{1}{2(5.6)(0.6)} + \frac{1}{(0.6)^2}\right)u_0 + \left(-\frac{2}{(0.6)^2} - \frac{1}{(5.6)^2}\right)u_1 + \left(\frac{1}{0.6^2} + \frac{1}{2(5.6)(0.6)}\right)u_2 = 0$$

$$2.6290u_0 - 5.5874u_1 + 2.9266u_2 = 0$$

Step 3 At node $i = 2,\quad r_2 = r_1 + \Delta r = 5.6 + 0.6 = 6.2$

$$\left(-\frac{1}{2(6.2)(0.6)} + \frac{1}{0.6^2}\right)u_1 + \left(-\frac{2}{0.6^2} - \frac{1}{6.2^2}\right)u_2 + \left(\frac{1}{0.6^2} + \frac{1}{2(6.2)(0.6)}\right)u_3 = 0$$

$$2.6434u_1 - 5.5816u_2 + 2.9122u_3 = 0$$

Solution Cont

Step 4 At node $i = 3, \quad r_3 = r_2 + \Delta r = 6.2 + 0.6 = 6.8$

$$\left(-\frac{1}{2(6.8)(0.6)} + \frac{1}{0.6^2}\right)u_2 + \left(-\frac{2}{0.6^2} - \frac{1}{6.8^2}\right)u_3 + \left(\frac{1}{0.6^2} + \frac{1}{2(6.8)(0.6)}\right)u_4 = 0$$

$$2.6552u_2 - 5.5772u_3 + 2.9003u_4 = 0$$

Step 5 At node $i = 4, \quad r_4 = r_3 + \Delta r = 6.8 + 0.6 = 7.4$

$$\left(-\frac{1}{2(7.4)(0.6)} + \frac{1}{0.6^2}\right)u_3 + \left(-\frac{2}{0.6^2} - \frac{1}{(7.4)^2}\right)u_4 + \left(\frac{1}{0.6^2} + \frac{1}{2(7.4)(0.6)}\right)u_5 = 0$$

$$2.6651u_3 - 5.6062u_4 + 2.8903u_5 = 0$$

Step 6 At node $i = 5, \quad r_5 = r_4 + \Delta r = 7.4 + 0.6 = 8$

$$u_5 = u/_{r=b} = 0.0030769$$

Solving system of equations

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2.6290 & -5.5874 & 2.9266 & 0 & 0 & 0 \\ 0 & 2.6434 & -5.5816 & 2.9122 & 0 & 0 \\ 0 & 0 & 2.6552 & -5.5772 & 2.9003 & 0 \\ 0 & 0 & 0 & 2.6651 & -5.6062 & 2.8903 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0.0038731 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.0030769 \end{bmatrix}$$

$u_0 = 0.0038731$ $\quad u_3 = 0.0032689$

$u_1 = 0.0036115$ $\quad u_4 = 0.0031586$

$u_2 = 0.0034159$ $\quad u_5 = 0.0030769$

Solution Cont

$$\left.\frac{du}{dr}\right|_{r=a} \approx \frac{u_2 - u_0}{2(\Delta r)} = \frac{0.0034159 - 0.0038731}{2(0.6)} = -0.00038100$$

$$\sigma_{\max} = \frac{30\times 10^6}{1-0.3^2}\left(\frac{0.0038731}{5} + 0.3(-0.000381)\right) = 21769\, psi$$

$$FS = \frac{36\times 10^3}{21769} = 1.6537$$

$$E_t = 20538 - 21769 = -1231.0$$

$$|\in_t| = \left|\frac{20538 - 21769}{20538}\right| \times 100 = 5.9938\ \%$$

Comparison of radial displacements

Table 1. Comparisons of radial displacements from two methods

r	u_{exact}	$u_{1st\ order}$	$\lvert\varepsilon_t\rvert$	$u_{2nd\ order}$	$\lvert\varepsilon_t\rvert$
5	0.0038731	0.0038731	0.0000	0.0038731	0.0000
5.6	0.0036110	0.0036165	1.5160×10^{-1}	0.0036115	1.4540×10^{-2}
6.2	0.0034152	0.0034222	2.0260×10^{-1}	0.0034159	1.8765×10^{-2}
6.8	0.0032683	0.0032743	1.8157×10^{-1}	0.0032689	1.6334×10^{-2}
7.4	0.0031583	0.0031618	1.0903×10^{-1}	0.0031586	9.5665×10^{-3}
8	0.0030769	0.0030769	0.0000	0.0030769	0.0000