Ordinary Differential Equations

Topic: Shooting Method

Major: General Engineering

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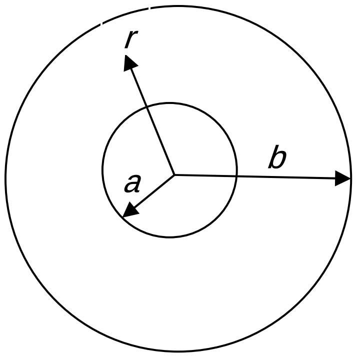
Shooting Method

The shooting method uses the methods used in solving initial value problems. This is done by assuming initial values that would have been given if the ordinary differential equation were a initial value problem. The boundary value obtained is compared with the actual boundary value. Using trial and error or some scientific approach, one tries to get as close to the boundary value as possible.

Example

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=0, u(5)=0.0038731,\ \ u(8)=0.0030769$$



Let

$$\frac{du}{dr}=w$$

Then

$$\frac{dw}{dr}+\frac{1}{r}w-\frac{u}{r^2}=0$$

Solution

Two first order differential equations are given as

$$\frac{du}{dr} = w, \quad u(5) = 0.0038371$$

$$\frac{dw}{dr} = -\frac{w}{r} + \frac{u}{r^2}, \quad w(5) = not\ known$$

Let us assume

$$w(5) = \frac{du}{dr}(5) \approx \frac{u(8) - u(5)}{8-5} = -0.00026540$$

To set up initial value problem

$$\frac{du}{dr} = w = f_1(r,u,w), u(5) = 0.0038371$$

$$\frac{dw}{dr} = -\frac{w}{r} + \frac{u}{r^2} = f_2(r,u,w), w(5) = -0.00026540$$

Solution Cont

Using Euler's method,

$$u_{i+1} = u_i + f_1(r_i, u_i, w_i)h$$

$$w_{i+1} = w_i + f_2(r_i, u_i, w_i)h$$

Let us consider 4 segments between the two boundaries, $r = 5$ and

$r = 8$ then,

$$h = \frac{8-5}{4} = 0.75$$

Solution Cont

For $i = 0, r_0 = 5, u_0 = 0.0038371, w_0 = -0.00026540$

$$u_1 = u_0 + f_1(r_0, u_0, w_0)h$$

$$= 0.0038371 + f_1(5, 0.0038371, -0.00026540)0.75$$

$$= 0.0038371 + (-0.00026540)0.75$$

$$= 0.0036741$$

$$w_1 = w_0 + f_2(r_0, u_0, w_0)h$$

$$= -0.00026540 + f_2(5, 0.0038371, -0.00026540)(0.75)$$

$$= -0.00026540 + \left(-\frac{-0.00026540}{5} + \frac{0.0038371}{5^2}\right)(0.75)$$

$$= -0.00010940$$

Solution Cont

For $i=1, r_1 = r_0 + h = 5 + 0.75 = 5.75,\ u_1 = 0.0036741, w_1 = -0.00010940$

$$u_2 = u_1 + f_1(r_1, u_1, w_1)h$$

$$= 0.0036741 + f_1(5.75, 0.0036741, -0.00010940)(0.75)$$

$$= 0.0036741 + (-0.00010940)0.75$$

$$= 0.0035920$$

$$w_2 = w_1 + f_2(r_1, u_1, w_1)h$$

$$= -0.0001094 + f_2(5.75, 0.0036741, -0.00010940)(0.75)$$

$$= -0.00010940 + (0.00013015)0.75$$

$$= -0.000011785$$

Solution Cont

For $i = 2, r_2 = r_1 + h = 5.75 + 0.75 = 6.5 \quad u_2 = 0.0035920, w_2 = -0.000011785$

$$u_3 = u_2 + f_1(r_2, u_2, w_2)h$$

$$= 0.0035920 + f_1(6.5, 0.0035920, -0.000011785)(0.75)$$

$$= 0.0035920 + (-0.000011785)0.75$$

$$= 0.0035832$$

$$w_3 = w_2 + f_2(r_2, u_2, w_2)h$$

$$= -0.000011785 + f_2(6.5, 0.0035920, -0.000011785)0.75$$

$$= -0.000011785 + (0.000086831)0.75$$

$$= 0.000053339$$

Solution Cont

For $i=3, r_3 = r_2 + h = 6.50 + 0.75 = 7.25 \quad u_3 = 0.0035832, w_3 = 0.000053339$

$$u_4 = u_3 + f_1(r_3, u_3, w_3)h$$
$$= 0.0035832 + f_1(7.25, 0.0035832, 0.000053339)0.75$$
$$= 0.0035832 + (0.000053339)0.75$$
$$= 0.0036232$$

$$w_4 = w_3 + f_2(r_3, u_3, w_3)h$$
$$= -0.000011785 + f_2(5.75, 0.0035920, -0.000011785)0.75$$
$$= 0.000053339 + (0.000060813)0.75$$
$$= 0.000098948$$

So at $r = r_4 = r_3 + h = 7.25 + 0.75 = 8$

$$u_4 = u(8) \cong 0.0036232$$

Solution Cont

Let us assume a new value for $\frac{du}{dr}(5)$

$$w(5) = \frac{du}{dr}(5) \cong 2\frac{u(8) - u(5)}{8 - 5} = 2(-0.0002654) = -0.00053080$$

Using $h = 0.75$ and Euler's method, we get

$$u_4 = u(8) \cong 0.0029664$$

While the given value of this boundary condition is

$$u_4 = u(8) = 0.0030769$$

Solution Cont

Using linear interpolation on the obtained data for the two assumed values of

$\frac{du}{dr}(5)$ we get

$$u(8) = 0.00030769$$

$$\frac{du}{dr}(5) \cong \frac{-0.00053080 - (-0.00026540)}{0.0029644 - 0.0036974}(0.0030769 - 0.0036974) + (-0.00026340)$$

$$= -0.00048614$$

Using $h = 0.75$ and repeating the Euler's method with $w(5) = -0.00048614$

$$u_4 = u(8) \cong 0.0030769$$

Solution Cont

Using linear interpolation to refine the value of u_4 till one gets close to the actual value of $u(8)$ which gives you,

$$u_1 = u(5) = 0.0038731$$

$$u_2 = u(5.75) \cong 0.0035085$$

$$u_3 = u(6.50) \cong 0.0032857$$

$$u_4 = u(7.25) \cong 0.0031517$$

$$u_5 = u(8.00) \cong 0.0030769$$

Comparisons of different initial guesses

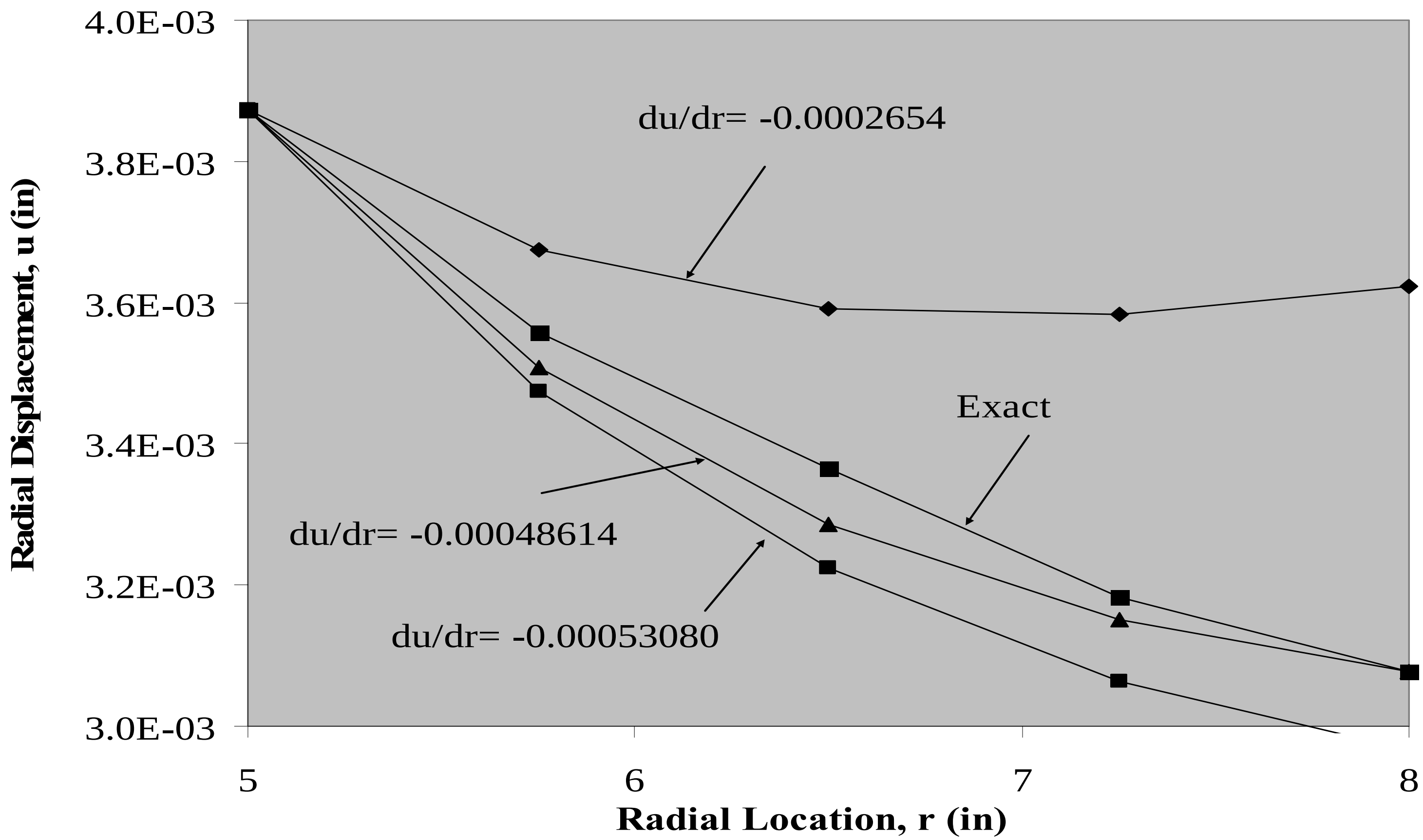


Figure 4. Comparision of results with different initial guesses of slope.

Comparison of Euler and Runge-Kutta Results with exact results

Table 1. Comparison of Euler and Runge-Kutta results with exact results.

r	Exact	Euler	$\|\varepsilon_t\|$ (%)	Runge-Kutta	$\|\varepsilon_t\|$ (%)
5	3.8731×10^{-3}	3.8731×10^{-3}	0.0000	3.8731×10^{-3}	0.0000
5.75	3.5567×10^{-3}	3.5085×10^{-3}	1.3731	3.5554×10^{-3}	3.5824×10^{-2}
6.5	3.3366×10^{-3}	3.2857×10^{-3}	1.5482	3.3341×10^{-3}	7.4037×10^{-2}
7.25	3.1829×10^{-3}	3.1517×10^{-3}	9.8967×10^{-1}	3.1792×10^{-3}	1.1612×10^{-1}
8	3.0770×10^{-3}	3.0769×10^{-3}	1.9500×10^{-3}	3.0723×10^{-3}	1.5168×10^{-1}