Fourier Transform Pair

Part: Frequency and Time Domain

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Lecture # 5

Chapter 11.03: Fourier Transform Pair: Frequency and Time Domain

Major: All Engineering Majors

Authors: Duc Nguyen

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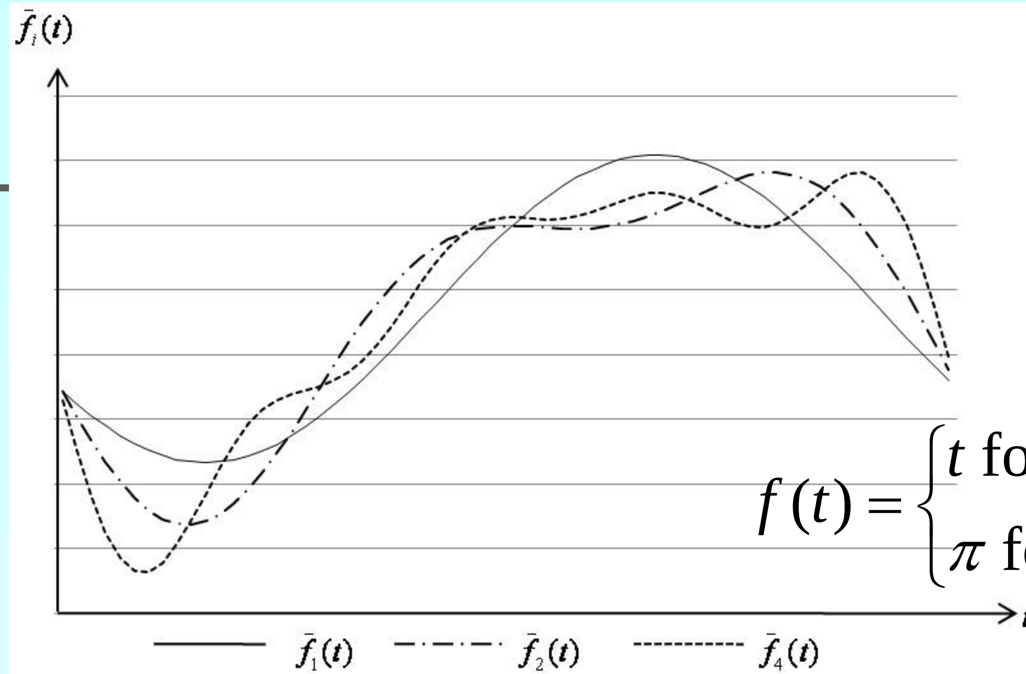
Numerical Methods for STEM undergraduates

Example 1



$$f(t)=\begin{cases} t \text{ for } 0<t\le\pi \\ \pi \text{ for } \pi\le t<2\pi(=T) \end{cases}$$

$$\bar{f}_1(t)\approx a_0+a_1Cos(t)+b_1Sin(t)$$

$$\bar{f}_2(t)\approx a_0+a_1Cos(t)+b_1Sin(t)+a_2Cos(2t)+b_2Sin(2t)$$

$$\bar{f}_4(t)\approx a_0+a_1Cos(t)+b_1Sin(t)+a_2Cos(2t)+b_2Sin(2t)$$
$$+a_3Cos(3t)+b_3Sin(3t)+a_4Cos(4t)+b_4Sin(4t)$$

Frequency and Time Domain

The amplitude (vertical axis) of a given periodic function can be plotted versus time (horizontal axis), but it can also be plotted in the frequency domain as shown in Figure 2.

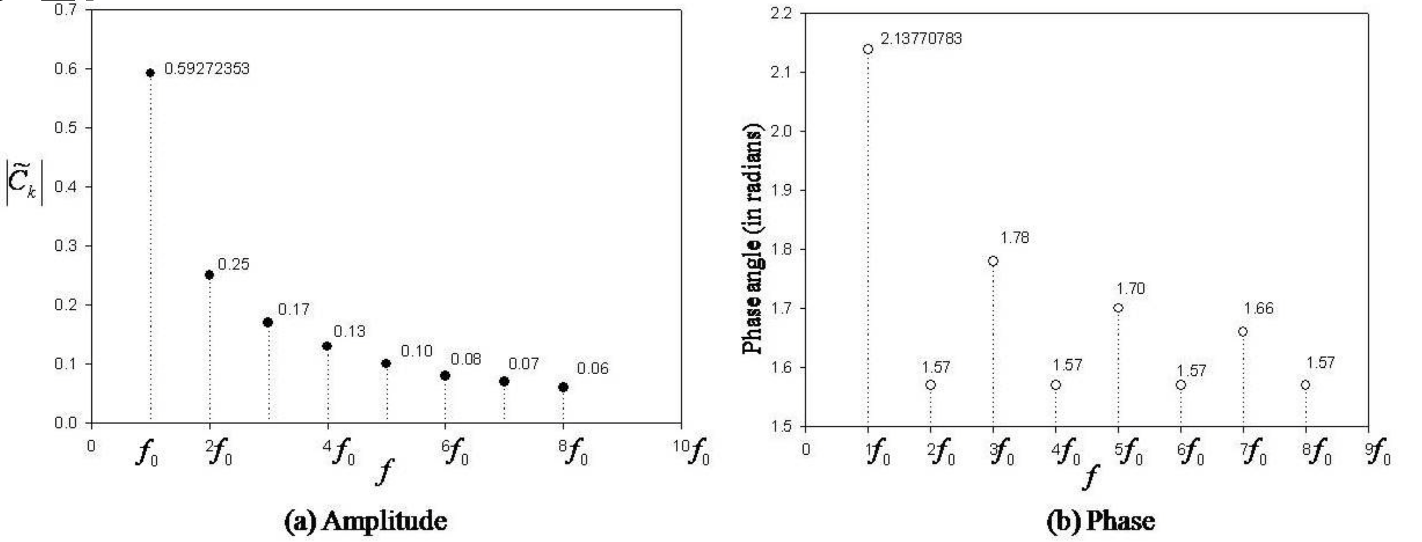


Figure 2 Periodic function (see Example 1 in Chapter 11.02 Continuous Fourier Series) in frequency domain.

Frequency and Time Domain cont.

Figures 2(a) and 2(b) can be described with the following equations from chapter 11.02,

$$f(t) = \sum_{k=-\infty}^{\infty} \widetilde{C}_k e^{ikw_0 t} \qquad \text{(39, repeated)}$$

where

$$\widetilde{C}_k = \left(\frac{1}{T}\right)\left\{\int_0^T f(t) \times e^{-ikw_0 t} dt\right\} \qquad \text{(41, repeated)}$$

Frequency and Time Domain cont.

For the periodic function shown in Example 1 of Chapter 11.02 (Figure 1), one has:

$$w_0 = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$\tilde{C}_k = \left(\frac{1}{T}\right)\left\{\int_0^{\pi} t \times e^{-ikt}dt + \int_{\pi}^{2\pi} \pi \times e^{-ikt}dt\right\}$$

Frequency and Time Domain cont.

Define:

$$A \equiv \int_0^{\pi} t \times e^{-ikt} dt = \left[t \times \left(\frac{-1}{ik} \right) e^{-ikt} \right]_0^{\pi} + \int_0^{\pi} \left(\frac{1}{ik} \right) e^{-ikt} dt$$

or

$$A = \left[\left(\frac{-\pi}{ik} \right) e^{-ik\pi} \right] + \left(\frac{1}{k^2} \right) \left[e^{-ik\pi} - 1 \right]$$

$$= \left[\left(\left(\frac{\pi i}{k} \right) e^{-ik\pi} + \left(\frac{1}{k^2} \right) e^{-ik\pi} - \frac{1}{k^2} \right) \right]$$

Frequency and Time Domain cont.

Also,

$$B \equiv \pi \int_{\pi}^{2\pi} e^{-ikt} dt = \left[\left(e^{-ikt} \right) \left(\frac{-\pi}{ik} \right) \right]_{\pi}^{2\pi}$$

$$B = \left(\frac{-\pi}{ik} \right) \left[e^{-ik2\pi} - e^{-ik\pi} \right] = \left(\frac{\pi i}{k} \right) \left[e^{-ik2\pi} - e^{-ik\pi} \right]$$

Frequency and Time Domain cont.

Thus:

$$\tilde{C}_k = \left(\frac{1}{2\pi}\right)\{A+B\}$$

$$\tilde{C}_k = \left(\frac{1}{2\pi}\right)\left\{e^{-ik\pi}\left(\frac{\pi i}{k}+\frac{1}{k^2}-\frac{\pi i}{k}\right)-\frac{1}{k^2}+\left(\frac{\pi i}{k}\right)e^{-ik2\pi}\right\}$$

Using the following Euler identities

$$e^{-ik\pi} = \cos(-k\pi)+i\sin(-k\pi)$$

$$= \cos(k\pi)-i\sin(k\pi)$$

$$= \cos(k\pi)$$

$$e^{-ik2\pi} = \cos(k2\pi)-i\sin(k2\pi) = \cos(k2\pi)$$

Frequency and Time Domain cont.

Noting that $\cos(k2\pi)=1$ for any integer k

$$\tilde{C}_k = \left(\frac{1}{2\pi}\right)\left\{Cos(k\pi)\times\left(\frac{1}{k^2}\right)-\frac{1}{k^2}+\left(\frac{\pi i}{k}\right)\underbrace{Cos(k2\pi)}_{1}\right\}$$

Frequency and Time Domain cont.

Also,

$$\cos(k\pi) = \begin{cases} -1 \; for \; k = odd \; number \; (=1,3,5,7,\ldots) \\ +1 \; for \; k = even \; number \; (=2,4,6,8,\ldots) \end{cases}$$

Thus,

$$\tilde{C}_k = \left(\frac{1}{2\pi}\right)\left\{\frac{(-1)^k}{k^2} - \frac{1}{k^2} + \frac{\pi i}{k}\right\}$$

$$\tilde{C}_k = \left(\frac{1}{2\pi k^2}\right)\left[(-1)^k - 1\right] + \left(\frac{1}{2k}\right)i$$

Frequency and Time Domain cont.

From Equation (36, Ch. 11.02), one has

$$\tilde{C}_k = \frac{a_k - ib_k}{2} \qquad \text{(36, repeated)}$$

Hence; upon comparing the previous 2 equations, one concludes:

$$a_k \equiv \left(\frac{1}{\pi k^2}\right)\left[(-1)^k - 1\right]$$

$$b_k = \left(\frac{-1}{k}\right)$$

Frequency and Time Domain cont.

For $k = 1,2,3,4...8;$ the values for a_k and b_k (based on the previous 2 formulas) are exactly identical as the ones presented earlier in Example 1 of Chapter 11.02.

Frequency and Time Domain cont.

Thus:

$$\tilde{C}_1 = \frac{a_1 - ib_1}{2} = \frac{\frac{-2}{\pi} - i(-1)}{2} = \frac{-1}{\pi} + \frac{1}{2}i$$

$$\tilde{C}_2 = \frac{a_2 - ib_2}{2} = \frac{0 - i\left(-\frac{1}{2}\right)}{2} = 0 + \frac{1}{4}i$$

Frequency and Time Domain cont.

$$\tilde{C}_3 = \frac{a_3 - ib_3}{2} = \frac{\left(\frac{-2}{9\pi}\right) - i\left(\frac{-1}{3}\right)}{2} = \left(\frac{-1}{9\pi}\right) + \frac{1}{6}i$$

$$\tilde{C}_4 = \frac{a_4 - ib_4}{2} = \frac{0 - i\left(\frac{-1}{4}\right)}{2} = 0 + \frac{1}{8}i$$

$$\tilde{C}_5 = \frac{a_5 - ib_5}{2} = \frac{\left(\frac{-2}{25\pi}\right) - i\left(\frac{-1}{5}\right)}{2} = \left(\frac{-1}{25\pi}\right) + \frac{1}{10}i$$

Frequency and Time Domain cont.

$$\tilde{C}_6 = \frac{a_6 - ib_6}{2} = \frac{0 - i\left(\frac{-1}{6}\right)}{2} = 0 + \frac{1}{12}i$$

$$\tilde{C}_7 = \frac{a_7 - ib_7}{2} = \frac{\left(\frac{-2}{49\pi}\right) - i\left(\frac{-1}{7}\right)}{2} = \left(\frac{-1}{49\pi}\right) + \frac{1}{14}i$$

$$\tilde{C}_8 = \frac{a_8 - ib_8}{2} = \frac{0 - i\left(\frac{-1}{8}\right)}{2} = 0 + \frac{1}{16}i$$

Frequency and Time Domain cont.

In general, one has

$$\tilde{C}_k = \begin{cases} \dfrac{-1}{k^2\pi} + \left(\dfrac{1}{2k}\right)i \;\; for\ k = 1,3,5,7,.. = odd\ number \\ \left(\dfrac{1}{2k}\right)i \;\; for\ k = 2,4,6,8,.. = even\ number \end{cases}$$

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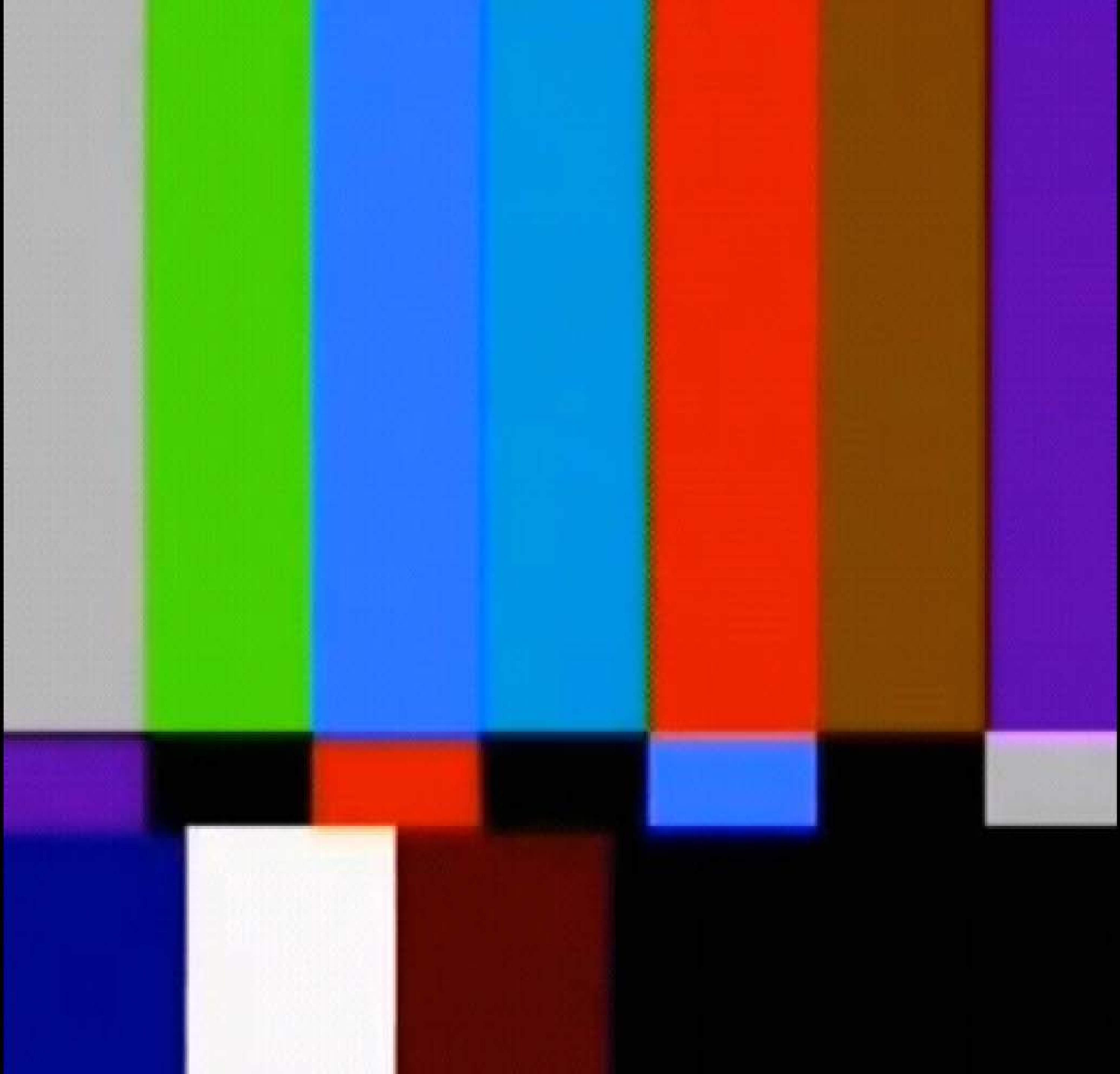

Fourier Transform Pair

Part: Complex Number in Polar Coordinates

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Lecture # 6

Chapter 11.03: Complex number in polar coordinates (Contd.)

In Cartesian (Rectangular) Coordinates, a complex number $\tilde{C}_k$ can be expressed as:

$$\tilde{C}_k = R_k + \left(I_k\right)i$$

In Polar Coordinates, a complex number $\tilde{C}_k$ can be expressed as:

$$\tilde{C}_k = Ae^{i\theta} = A\left\{\cos(\theta) + i\sin(\theta)\right\} = \left\{A\cos(\theta)\right\} + \left\{A\sin(\theta)\right\}i$$

Complex number in polar coordinates cont.

Thus, one obtains the following relations between the Cartesian and polar coordinate systems:

$$R_k = A\cos(\theta) \quad I_k = A\sin(\theta)$$

This is represented graphically in Figure 3.

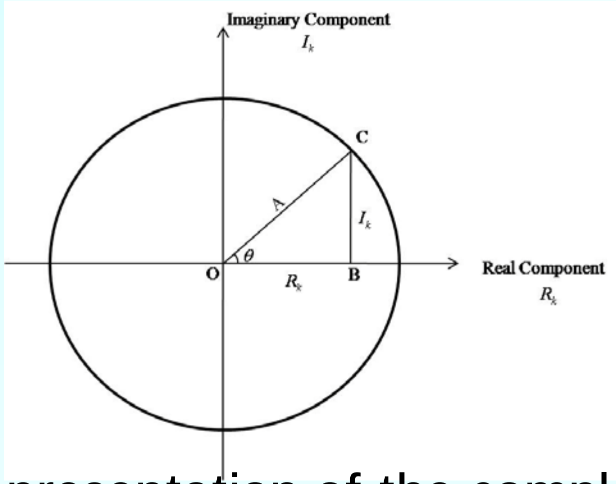


Figure 3. Graphical representation of the complex number system in polar coordinates.

Complex number in polar coordinates cont.

Hence

$$R_k^2 + I_k^2 = A^2\cos^2(\theta) + A^2\sin^2(\theta) = A^2\left[\cos^2(\theta) + \sin^2(\theta)\right]$$

$$\cos(\theta) = \frac{R_k}{A} \Rightarrow \theta = \cos^{-1}\left(\frac{R_k}{A}\right)$$

and

$$\sin(\theta) = \frac{I_k}{A} \Rightarrow \theta = \sin^{-1}\left(\frac{I_k}{A}\right)$$

Complex number in polar coordinates cont.

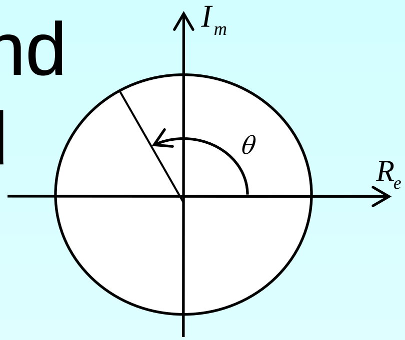
Based on the above 3 formulas, the complex numbers $\tilde{C}_k$ can be expressed as:

$$\tilde{C}_1 = \frac{-1}{\pi} + \left(\frac{1}{2}\right)i = (0.59272353)e^{i(2.13770783)}$$

Complex number in polar coordinates cont.

Notes:

(a) The amplitude and angle $\tilde{C}_1$ are 0.59 and 2.14 respectively (also see Figures 2a, and 2b in chapter 11.03).



(b) The angle θ (in radian) obtained from

$$Cos(\theta) = \frac{R_k}{A}$$ will be 2.138 radians (=122.48º).

However based on $$Sin(\theta) = \frac{I_k}{A}$$

Then θ = 1.004 radians (=57.52º).

Complex number in polar coordinates cont.

Since the Real and Imaginary components of θ are negative and positive, respectively, the proper selection for θ should be 2.1377 radians.

$$\tilde{C}_2 = 0 + \frac{1}{4}i = (0.25)e^{i\left(\frac{\pi}{2}\right)} = (0.25)e^{i(1.57079633)}$$

$$\tilde{C}_3 = \left(\frac{-1}{9\pi}\right) + \frac{1}{6}i = (0.17037798)e^{i(1.77990097)}$$

Complex number in polar coordinates cont.

$$\tilde{C}_4 = 0 + \frac{1}{8} i = (0.125) e^{i\left(\frac{\pi}{2}\right)} = (0.125) e^{i(1.57079633)}$$

$$\tilde{C}_5 = \left(\frac{-1}{25\pi}\right) + \frac{1}{10} i = (0.100807311) e^{i(1.69743886)}$$

$$\tilde{C}_6 = 0 + \frac{1}{12} i = (0.08333333) e^{i\left(\frac{\pi}{2}\right)} = (0.08333333) e^{i(1.57079633)}$$

Complex number in polar coordinates cont.

$$\tilde{C}_7 = \left(\frac{-1}{49\pi}\right) + \frac{1}{14}i = (0.07172336)e^{i(1.66149251)}$$

$$\tilde{C}_8 = 0 + \frac{1}{16}i = (0.0625)e^{i\left(\frac{\pi}{2}\right)}$$

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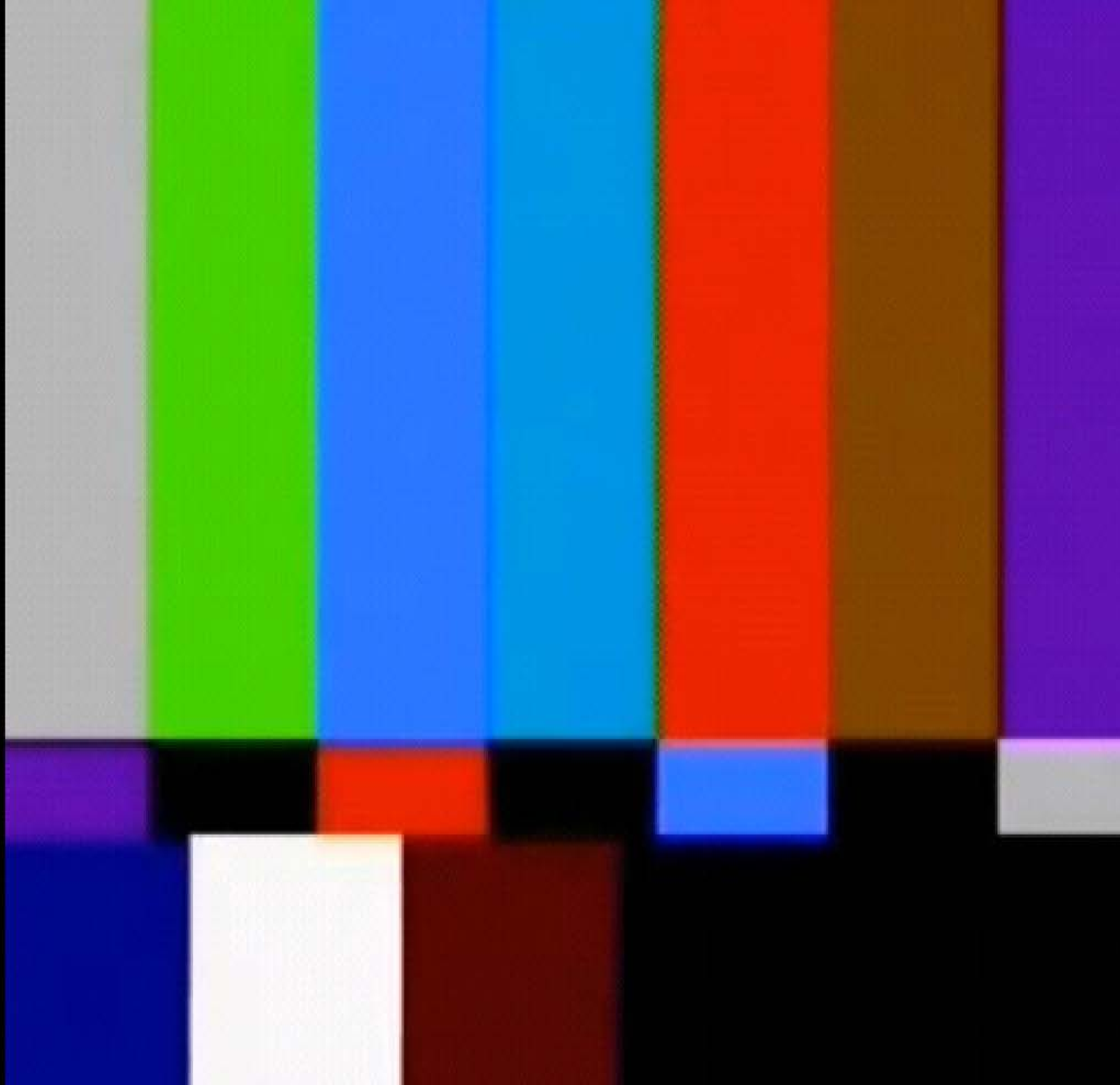

Numerical Methods

Fourier Transform Pair

Part: Non-Periodic Functions

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Lecture # 7

Chapter 11. 03: Non-Periodic Functions (Contd.)

Recall

$$f(t) = \sum_{k=-\infty}^{\infty} \tilde{C}_k e^{ikw_0 t} \qquad (39, \text{ repeated})$$

$$\tilde{C}_k = \left(\frac{1}{T}\right)\left\{\int_0^T f(t) \times e^{-ikw_0 t} dt\right\} \qquad (41, \text{ repeated})$$

Define

$$\hat{F}(ikw_0) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-ikw_0 t} dt \qquad (1)$$

Non-Periodic Functions

Then, Equation (41) can be written as

$$\tilde{C}_k = \left(\frac{1}{T}\right) \times \hat{F}(ikw_0)$$

And Equation (39) becomes

$$f(t) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{T}\right) \times \hat{F}(ikw_0) e^{ikw_0 t}$$

From above equation

$$f_{np}(t) = \lim_{\substack{T \to \infty \\ or \Delta f \to 0}} f(t) = \lim_{\Delta f \to 0} \sum_{k=-\infty}^{\infty} (\Delta f) \times \hat{F}(ikw_0) e^{ikw_0 t}$$

or

$$f_{np}(t) = \lim_{\Delta f \to 0} \sum_{k=-\infty}^{\infty} (\Delta f) \times \hat{F}(ik 2\pi \Delta f) e^{ik 2\pi \Delta f t}$$

Non-Periodic Functions cont.

From Figure 4,

$$k\Delta f = f$$

$$f_{np}(t) = \int df \times \hat{F}(i2\pi f)e^{i2\pi ft}$$

$$f_{np}(t) = \int \hat{F}(i2\pi f)e^{i2\pi ft} df$$

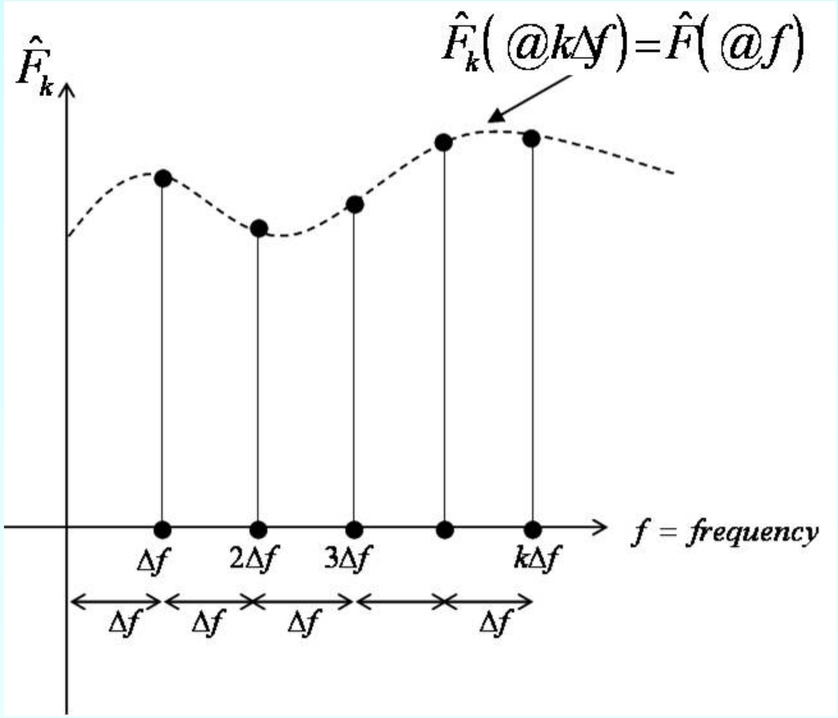

Figure 4. Frequency are discretized.

Non-Periodic Functions cont.

Multiplying and dividing the right-hand-side of the equation by 2π , one obtains

$$f_{np}(t) = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} \hat{F}(iw_0)e^{iw_0 t}d(w_0)$$; <u>inverse Fourier transform</u>

Also, using the definition stated in Equation (1), one gets

$$\hat{F}(iw_0) = \int_{-\infty}^{\infty} f_{np}(t)e^{-iw_0 t}d(t)$$; <u>Fourier transform</u>

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