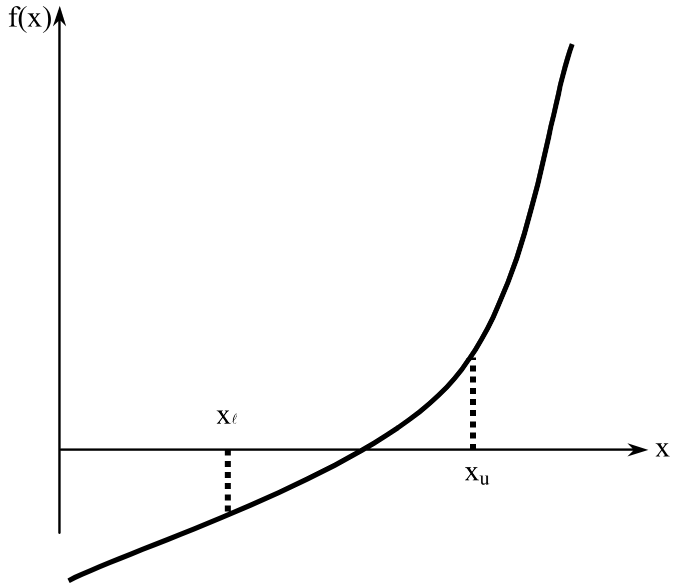
Roots of a Nonlinear Equation

Topic: Bisection Method

Major: Chemical Engineering

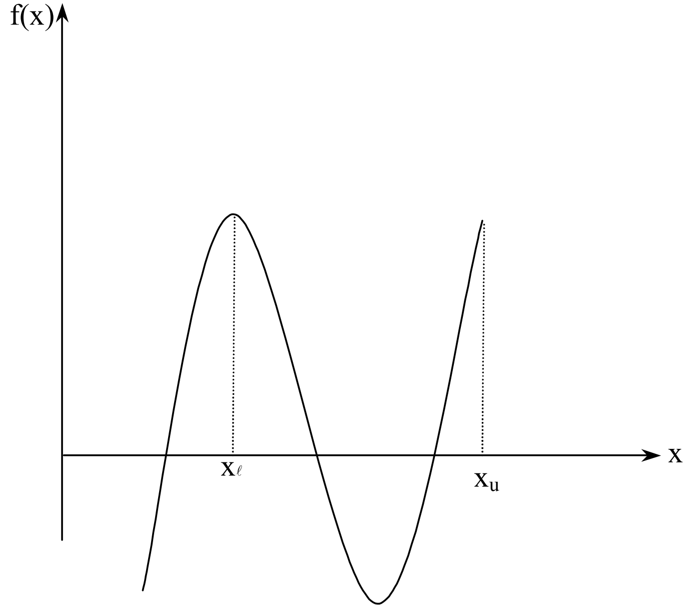
Basis of Bisection Method

Theorem: An equation f(x)=0, where f(x) is a real continuous function, has at least one root between x_ℓ and x_u if $f(x_l)\, f(x_u) < 0$.



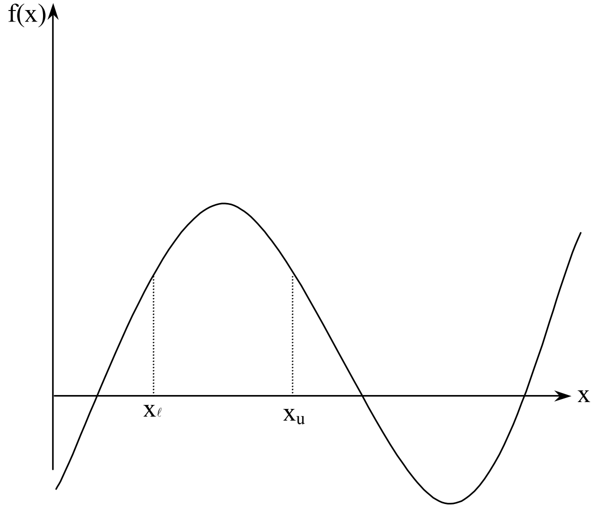
Theorem

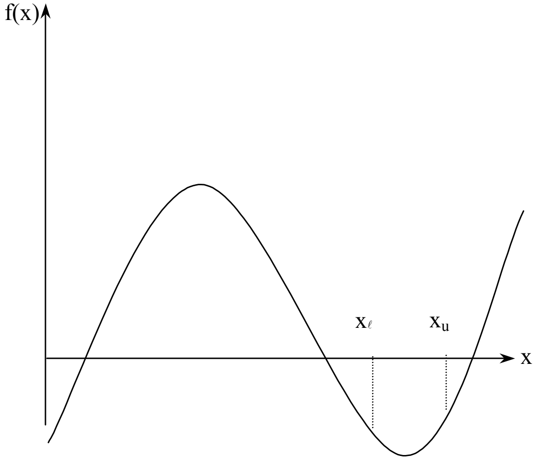
If function f(x) in f(x)=0 does not change sign between two points, roots may still exist between the two points.



Theorem

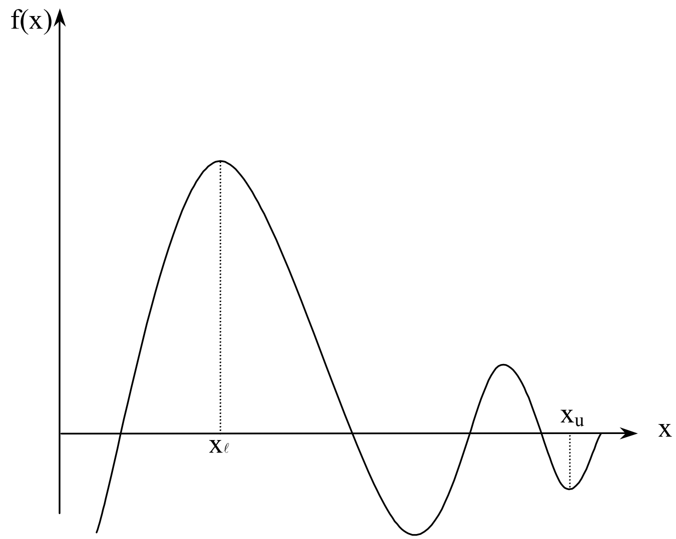
If the function f(x) in f(x)=0 does not change sign between two points, there may not be any roots between the two points.





Theorem

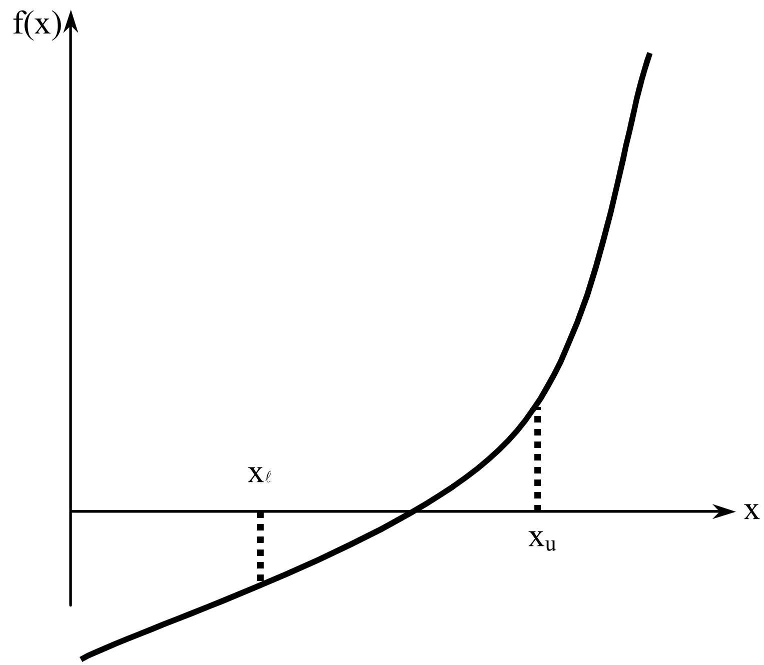
If the function f(x) in f(x)=0 changes sign between two points, more than one root may exist between the two points.



Algorithm for Bisection Method

Step 1

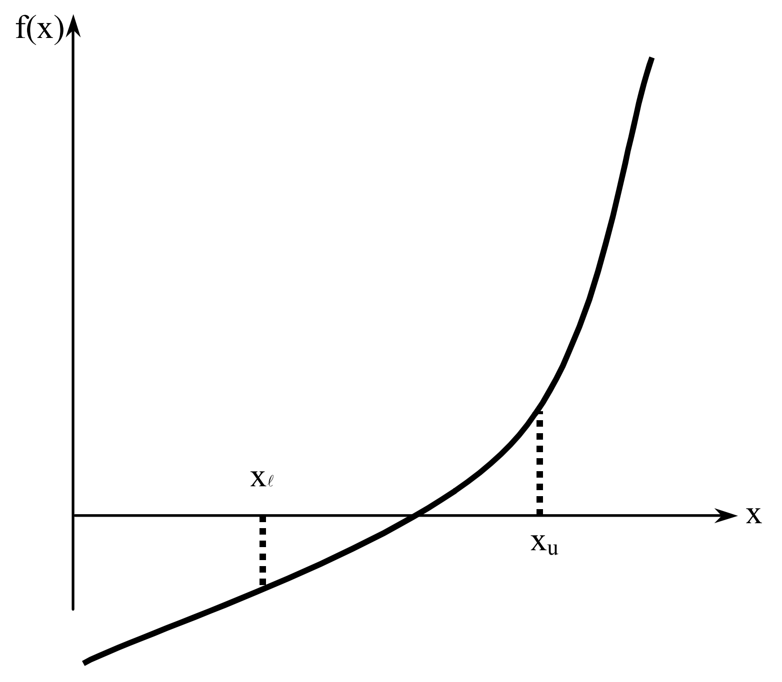
- Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)\,f(x_u) < 0$, or in other words, f(x) changes sign between x_R and x_u.



Step 2

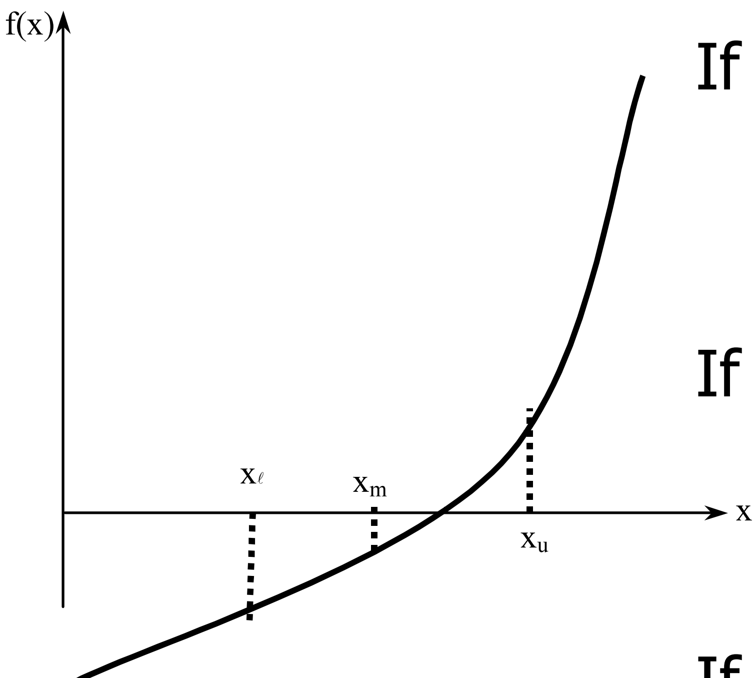
Estimate the root, x_m of the equation f (x) = 0 as the mid-point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$



Step 3

Now check the following



If $f(x_\ell)\, f(x_m) < 0$, then the root lies between x_R and x_m; then $x_\ell = x_\ell$; $x_u = x_m$.

If $f(x_R\ )\, f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\ell = x_m$; $x_u = x_u$.

If $f(x_\ell)\, f(x_m) = 0$; then the root is $x_{m.}$ Stop the algorithm if this is true.

Step 4

New estimate

$$x_m = \frac{x_\ell + x_u}{2}$$

Absolute Relative Approximate Error

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

$$x_m^{old} = \text{previous estimate of root}$$

$$x_m^{new} = \text{current estimate of root}$$

Step 5

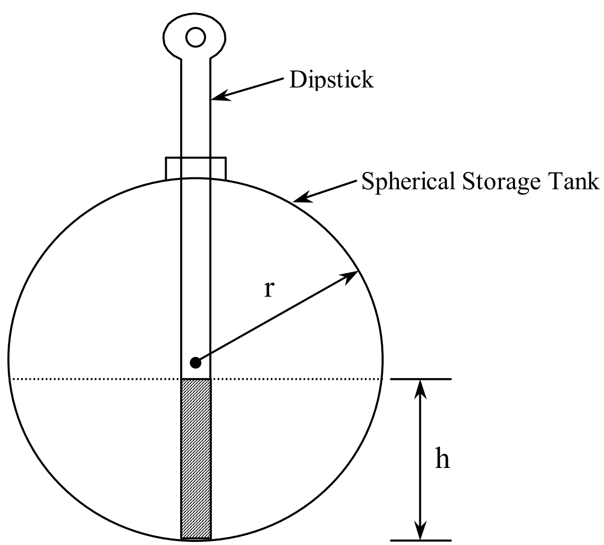
Check if absolute relative approximate error is less than prespecified tolerance or if maximum number of iterations is reached.

Yes → Stop

No → Using the new upper and lower guesses from Step 3, go to Step 2.

Example

- You have a spherical storage tank containing oil. The tank has a diameter of 6 ft. You are asked to calculate the height 'h' to which a dipstick 8 ft long would be wet with oil when immersed in the tank when it contains 4 ft^3 of oil.

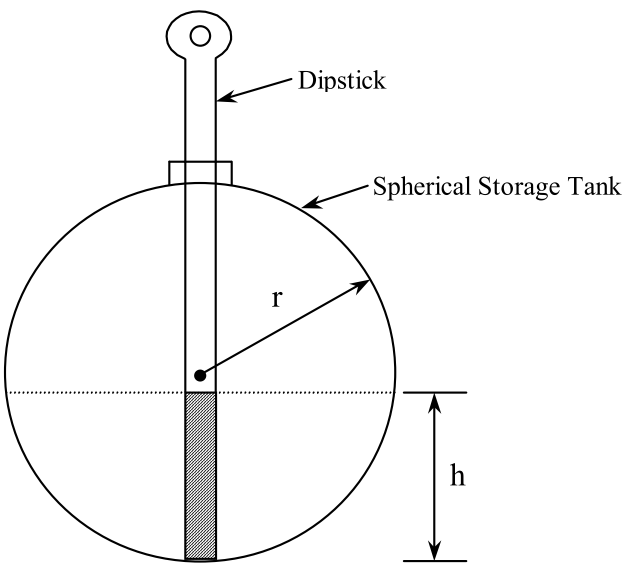


Solution

The equation that gives the height 'h' of liquid in the spherical tank for the given volume and radius is given by:

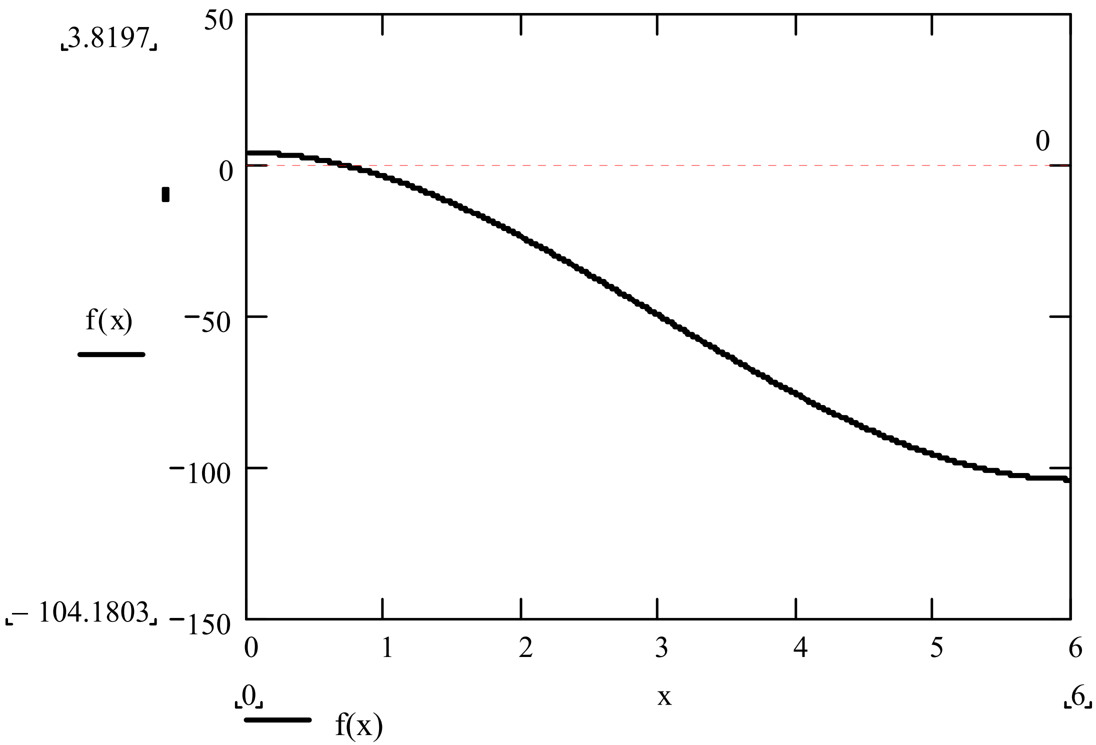
$$f(h) = h^3 - 9h^2 + 3.8197 = 0$$

Use the bisection method of finding roots of equations to find the depth 'h' to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation.

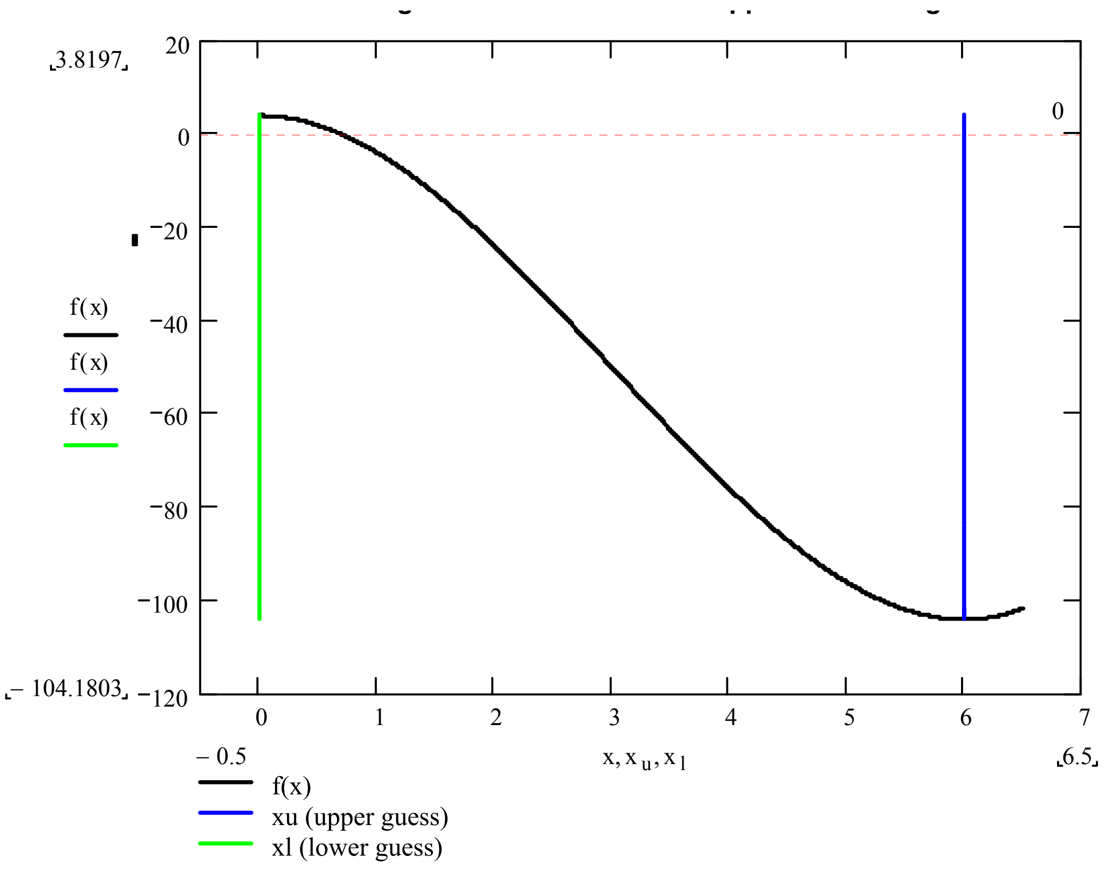


Graph of function f(x)

$$f(h) = h^3 - 9h^2 + 3.8197 = 0$$



Checking if the bracket is valid



Choose the bracket

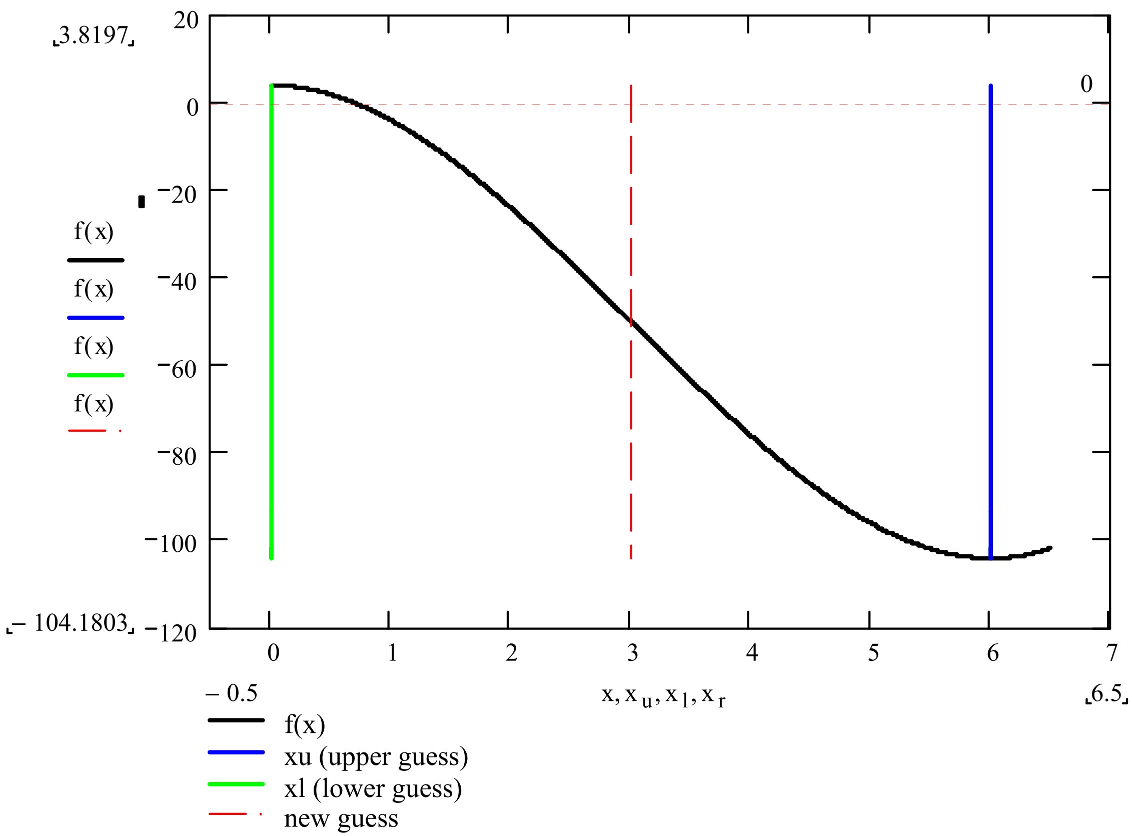
$$h_\ell = 0$$

$$h_u = 6$$

$$f(0.0) = 3.8197$$

$$f(6) = -104.1803$$

Iteration #1



$$h_\ell = 0, h_u = 6$$

$$h_m = \frac{0+6}{2} = 3$$

$$f(0) = 3.8197$$

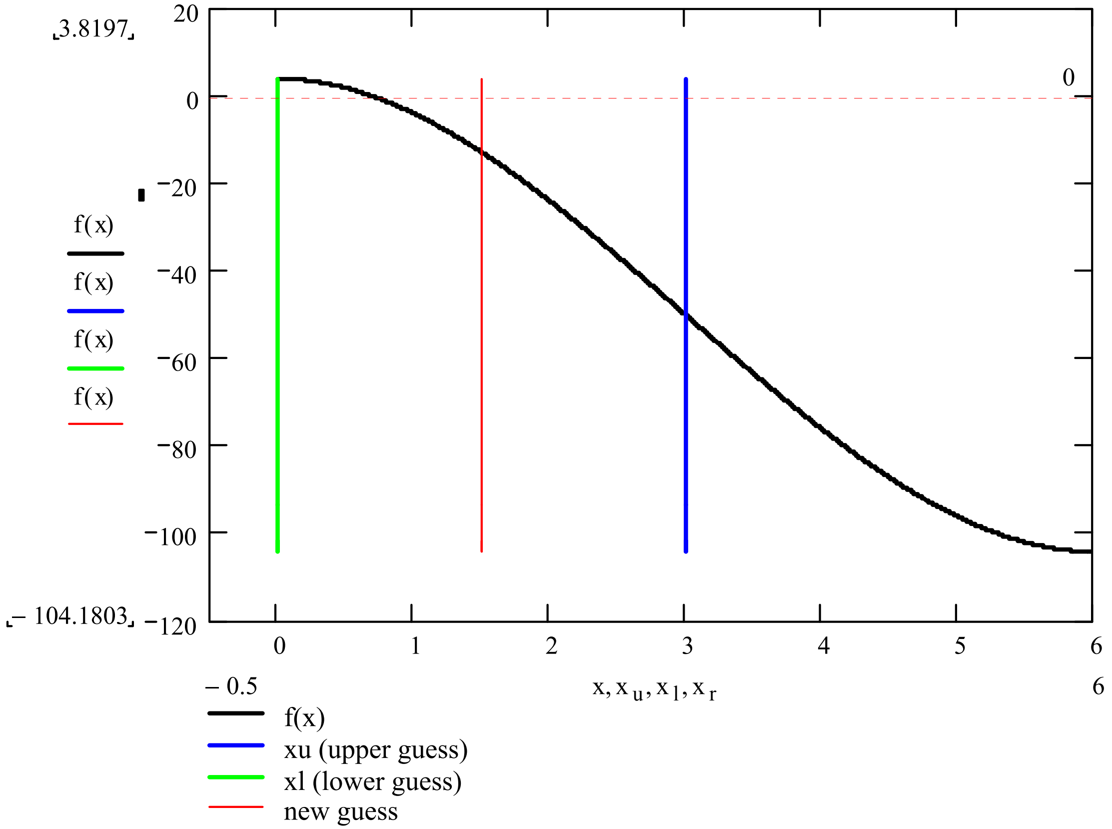
$$f(6) = -104.1803$$

$$f(3) = -50.18$$

$$h_\ell = 0$$

$$h_u = 3$$

Iteration #2



$$h_\ell = 0, h_u = 3$$

$$h_m = \frac{0+3}{2} = 1.5$$

$$|\in_a| = 100\%$$

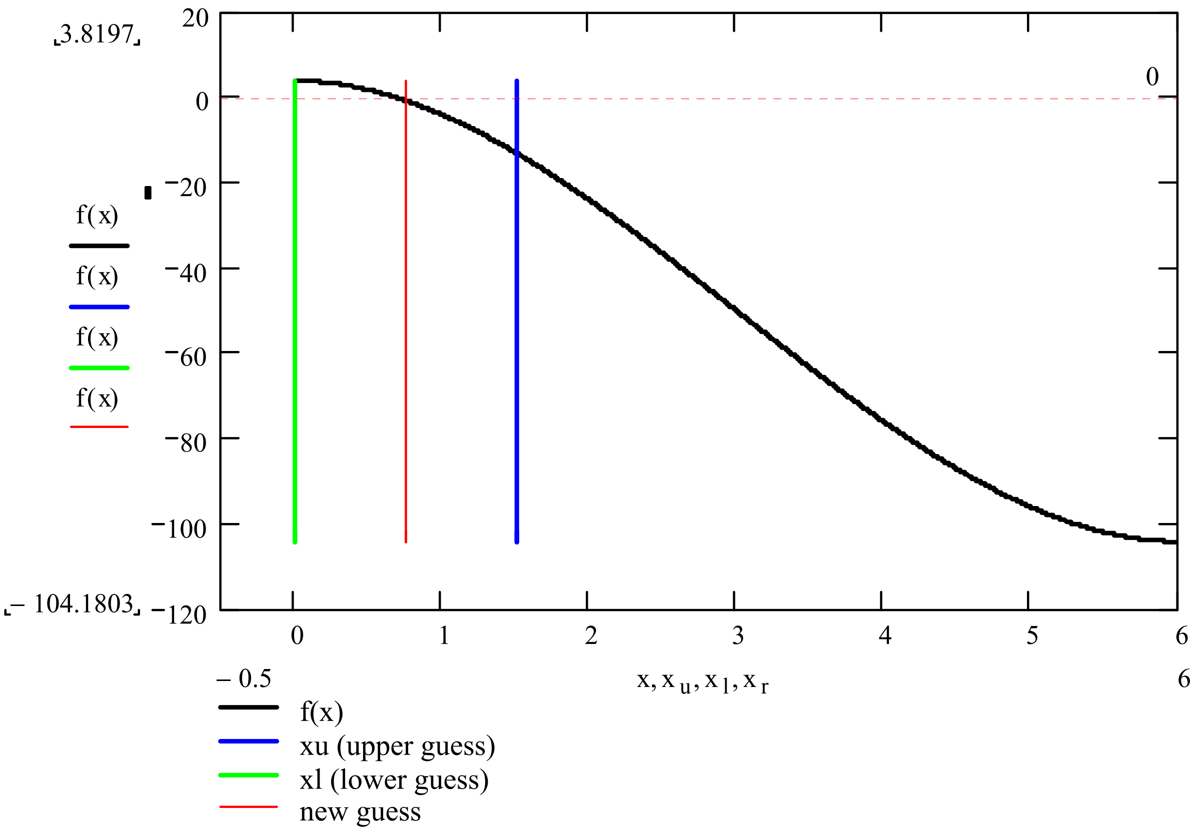
$$f(0) = 3.8197$$

$$f(3) = -50.18$$

$$f(1.5) = -13.0553$$

$$h_\ell = 0, h_u = 1.5$$

Iteration #3



$$h_\ell = 0, h_u = 1.5$$

$$h_m = \frac{0+1.5}{2} = 0.75$$

$$|\epsilon_a| = 100\%$$

$$f(0) = 3.8197$$

$$f(1.5) = -13.0553$$

$$f(0.75) = -0.8209$$

Convergence

Table 1: Root of f(x)=0 as function of number of iterations for bisection method.

Iteration	h_l	h_u	h_m	$\lvert \in_a \rvert$ %	$f(h_m)$
1	0.00	6	3	----------	-50.18
2	0.00	3	1.5	100	-13.0553
3	0.00	1.5	0.75	100	-0.8209
4	0.00	0.75	0.375	100	2.6068
5	0.375	0.75	0.5625	33.33	1.1500
6	0.5625	0.75	0.6563	14.2857	0.2263
7	0.6563	0.75	0.7031	6.6667	-0.2821
8	0.6563	0.7031	0.6797	3.4483	-0.0241
9	0.6563	0.6797	0.6680	1.7544	0.1021
10	0.6680	0.6797	0.6738	0.8696	0.0392

Advantages

- Always convergent
- The root bracket gets halved with each iteration - guaranteed.

Drawbacks

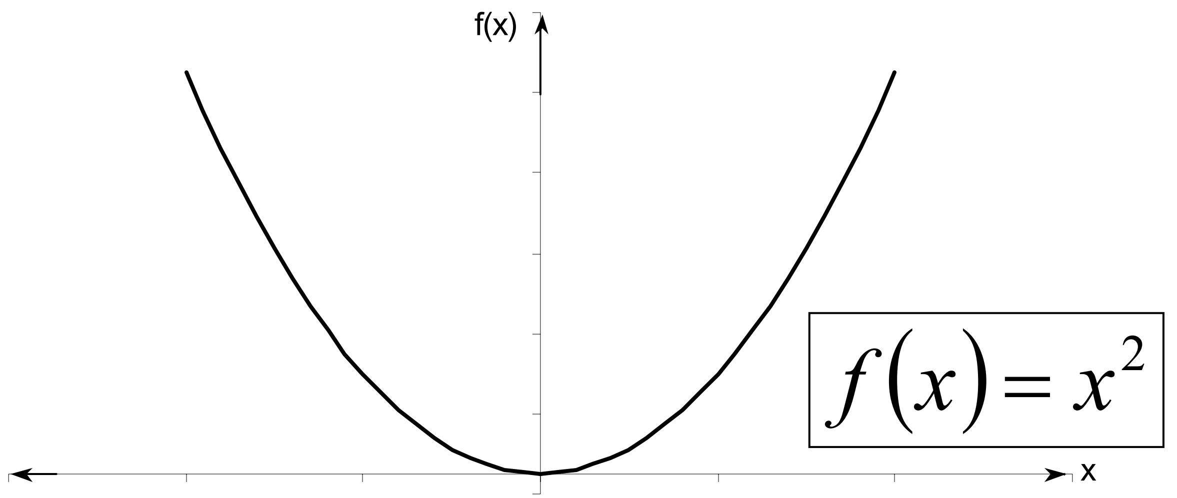
- Slow convergence

Drawbacks (continued)

- If one of the initial guesses is close to the root, the convergence is slower

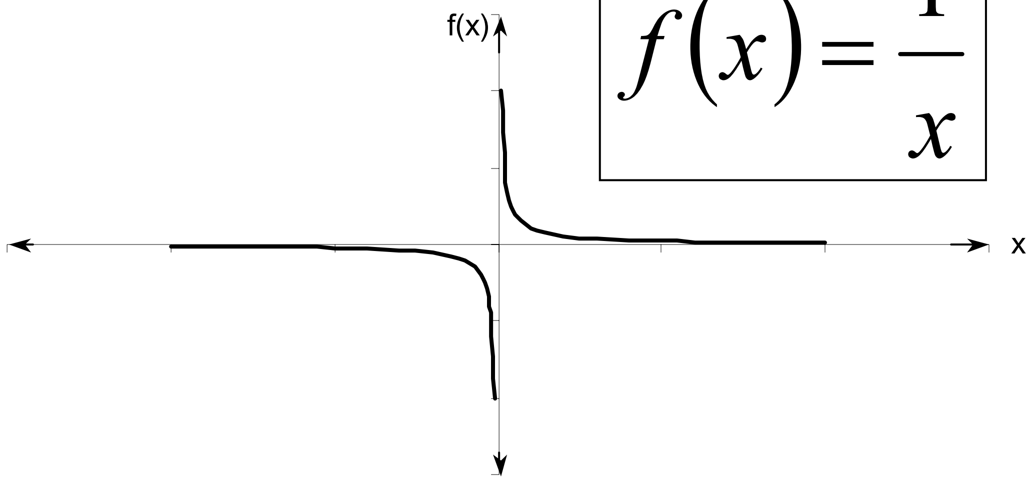
Drawbacks (continued)

- If a function f(x) is such that it just touches the x-axis it will be unable to find the lower and upper guesses.



Drawbacks (continued)

- Function changes sign but root does not exist



$$f(x) = \frac{1}{x}$$