Holistic Numerical Methods Institute

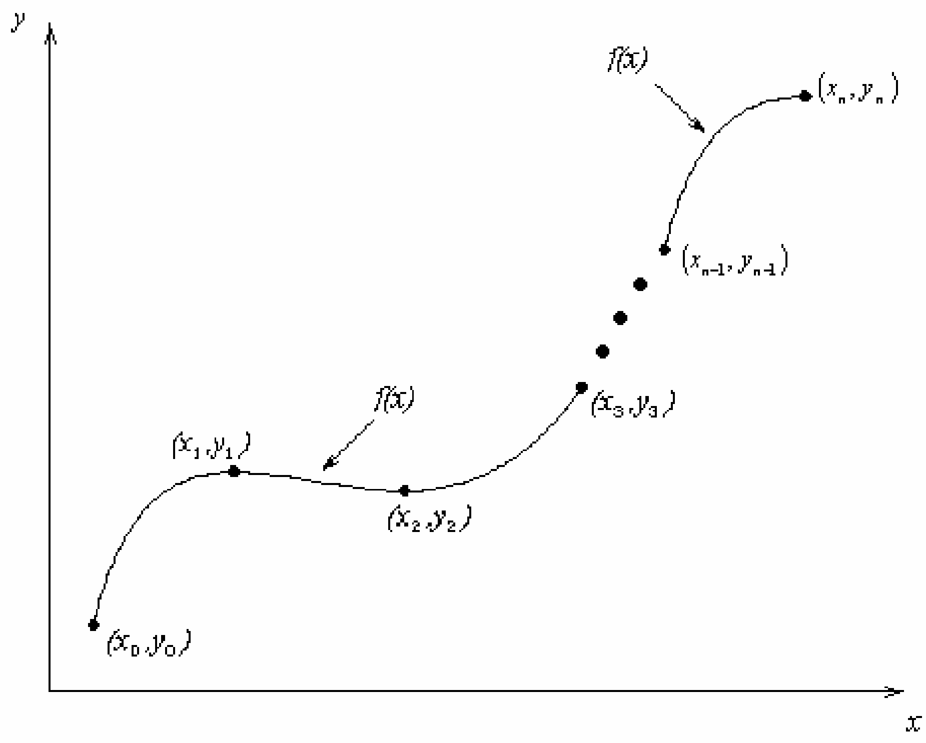
Interpolation

Topic: Lagrangian Interpolation

Major: Chemical

What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where ' n ' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

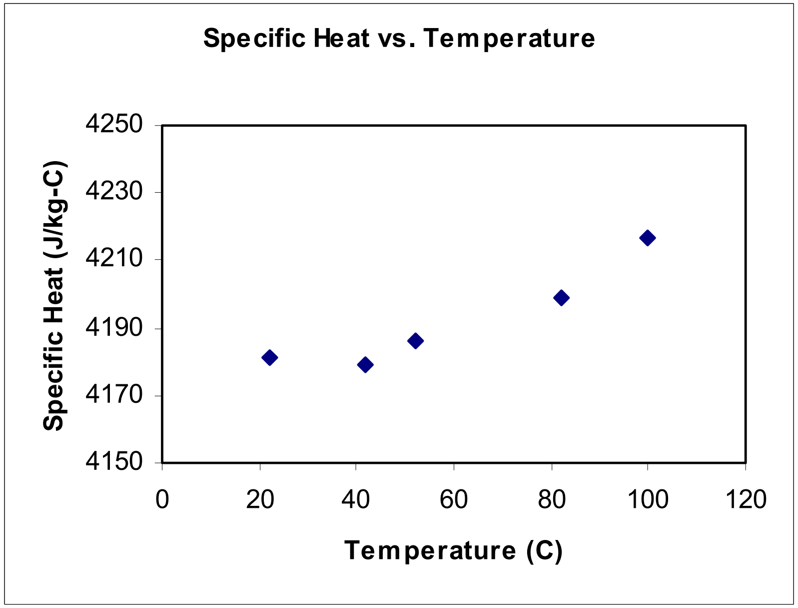
$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Example

To find how much heat is required to bring a kettle of water to boiling point, you are asked to calculate the specific heat of water at 61^0C. Use linear, quadratic and cubic interpolation.

Temperature	Specific heat
°C	$\frac{J}{kg - ^\circ C}$
22	4181
42	4179
52	4186
82	4199
100	4217



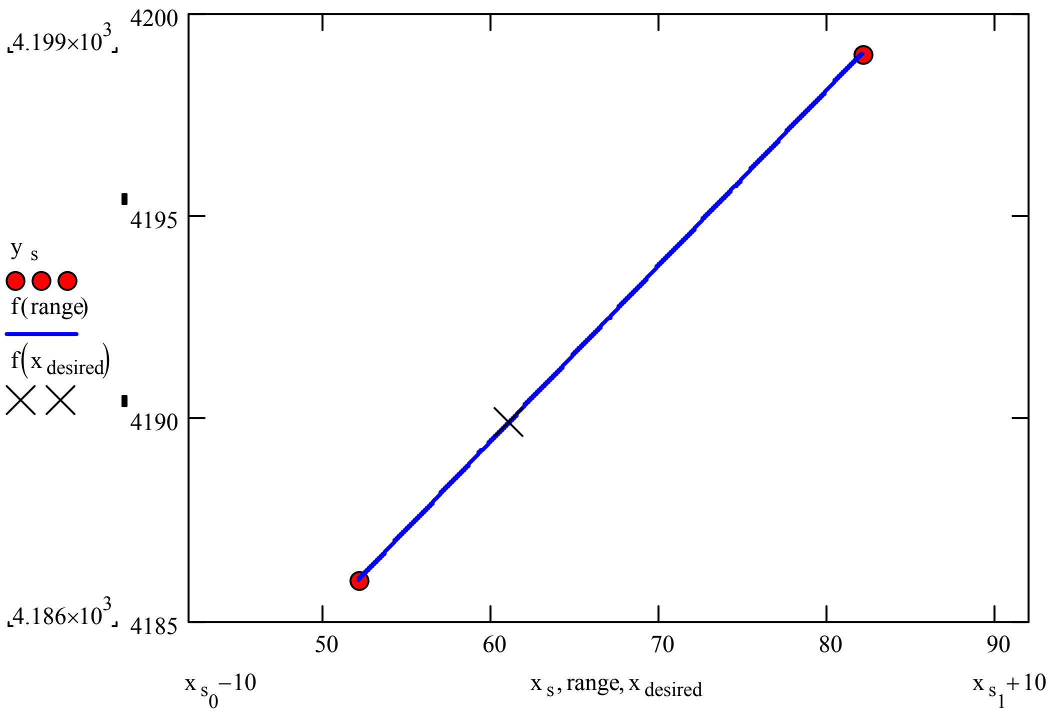
Linear Interpolation

$$C_p(T) = \sum_{i=0}^{1} L_i(T) C_p(T_i)$$

$$= L_0(T) C_p(T_0) + L_1(T) C_p(T_1)$$

$$T_0 = 52, C_p(T_0) = 4186$$

$$T_1 = 82, C_p(T_1) = 4199$$



Linear Interpolation (contd)

$$L_0(T) = \prod_{\substack{j=0 \\ j\neq 0}}^{1} \frac{T - T_j}{T_0 - T_j} = \frac{T - T_1}{T_0 - T_1}$$

$$L_1(T) = \prod_{\substack{j=0 \\ j\neq 1}}^{1} \frac{T - T_j}{T_1 - T_j} = \frac{T - T_0}{T_1 - T_0}$$

$$C_p(T) = \frac{T - T_1}{T_0 - T_1} C_p(T_0) + \frac{T - T_0}{T_1 - T_0} C_p(T_1) = \frac{T - 82}{52 - 82}(4186) + \frac{T - 52}{82 - 52}(4199)$$

$$C_p(61) = \frac{61 - 82}{52 - 82}(4186) + \frac{61 - 52}{82 - 52}(4199) = 0.7(4186) + 0.3(4199)$$

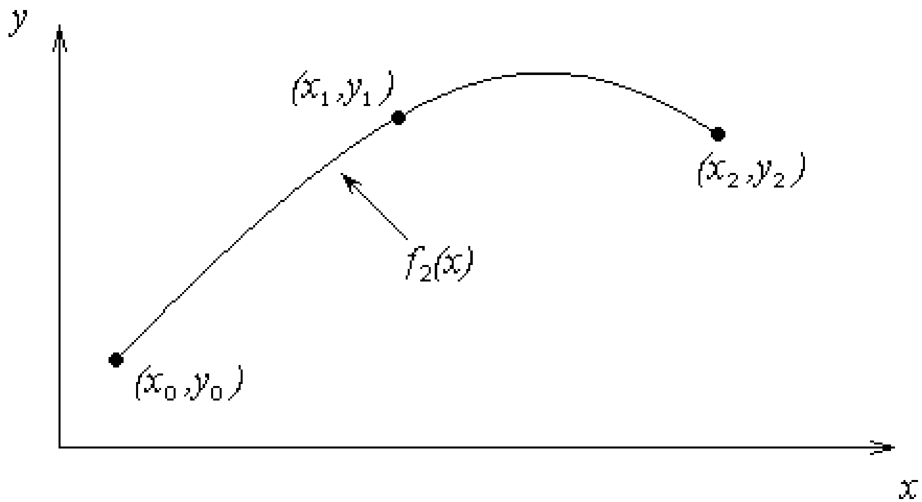
$$= 4189.9 \frac{J}{kg - ^\circ C}$$

Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the specific heat given by

$$C_p(T) = \sum_{i=0}^{2} L_i(T) C_p(T_i)$$

$$= L_0(T)C_p(T_0) + L_1(T)C_p(T_1) + L_2(T)C_p(T_2)$$



Quadratic Interpolation (contd)

$$T_o = 42,\ \ C_p(T_o) = 4179$$

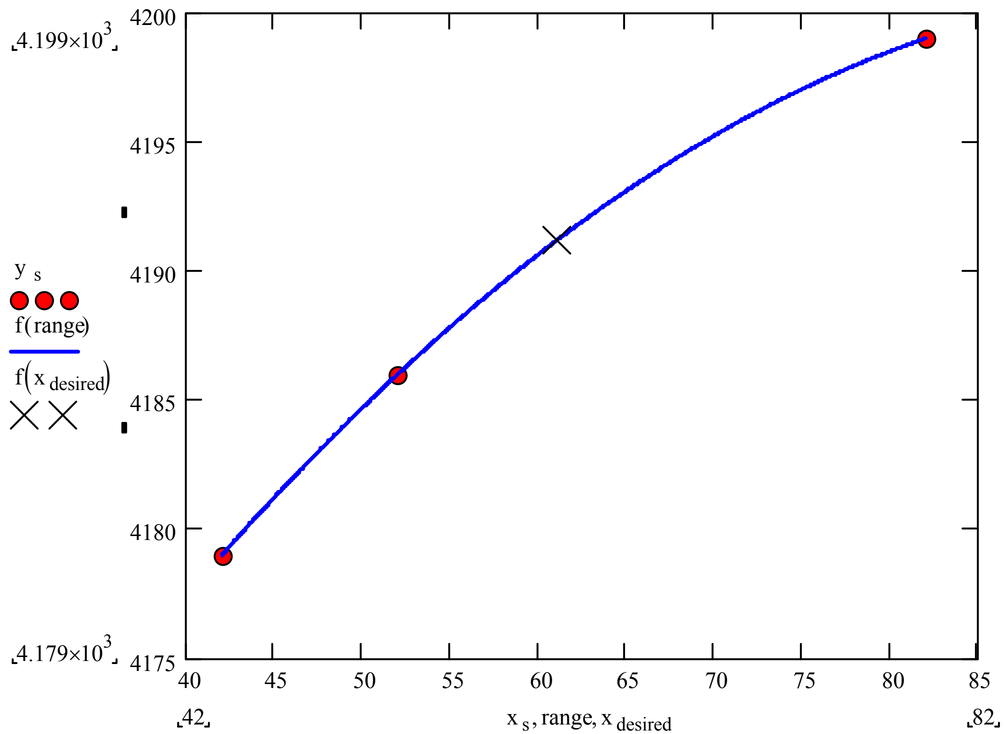
$$T_1 = 52,\ \ C_p(T_1) = 4186$$

$$T_2 = 82,\ \ C_p(T_2) = 4199$$

$$L_0(T) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{T - T_j}{T_0 - T_j} = \left(\frac{T - T_1}{T_0 - T_1}\right)\left(\frac{T - T_2}{T_0 - T_2}\right)$$

$$L_1(T) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{T - T_j}{T_1 - T_j} = \left(\frac{T - T_0}{T_1 - T_0}\right)\left(\frac{T - T_2}{T_1 - T_2}\right)$$

$$L_2(T) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{T - T_j}{T_2 - T_j} = \left(\frac{T - T_0}{T_2 - T_0}\right)\left(\frac{T - T_1}{T_2 - T_1}\right)$$



Quadratic Interpolation (contd)

$$C_p(T) = \left(\frac{T-T_1}{T_0-T_1}\right)\left(\frac{T-T_2}{T_0-T_2}\right)C_p(T_0) + \left(\frac{T-T_0}{T_1-T_0}\right)\left(\frac{T-T_2}{T_1-T_2}\right)C_p(T_1) + \left(\frac{T-T_0}{T_2-T_0}\right)\left(\frac{T-T_1}{T_2-T_1}\right)C_p(T_2)$$

$$C_p(61) = \frac{(61-52)(61-82)}{(42-52)(42-82)}(4179) + \frac{(61-42)(61-82)}{(52-42)(52-82)}(4186) + \frac{(61-42)(61-52)}{(82-42)(82-52)}(4199)$$

$$= (-0.4725)(4179) + (1.33)(4186) + (0.1425)(4199)$$

$$= 4191.2 \frac{J}{kg - ^\circ C}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{4191.2 - 4189.9}{4191.2}\right| \times 100$$

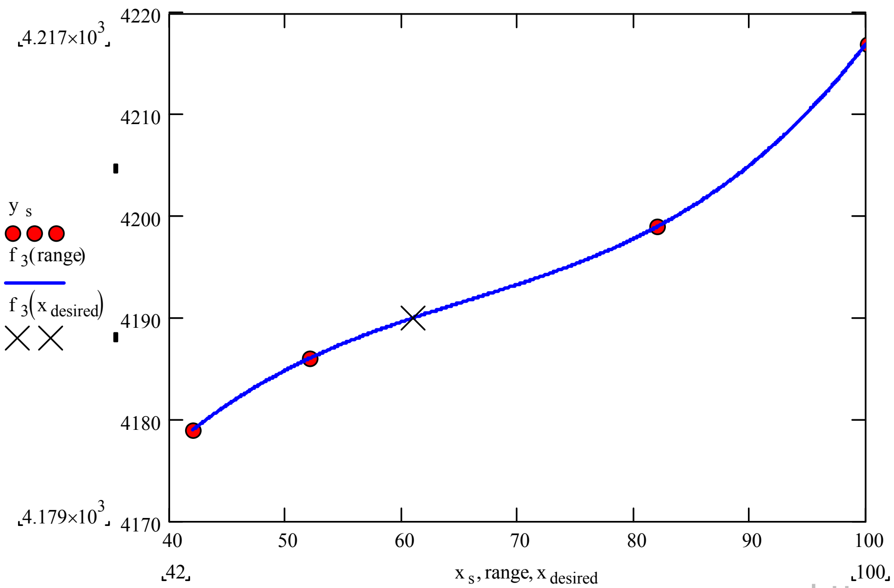
$$= 0.031017\%$$

Cubic Interpolation

For the third order polynomial (also called cubic interpolation), we choose the specific heat given by

$$C_p(T) = \sum_{i=0}^{3} L_i(T) C_p(T_i)$$

$$= L_0(T)C_p(T_0) + L_1(T)C_p(T_1) + L_2(T)C_p(T_2) + L_3(T)C_p(T_3)$$



Cubic Interpolation (contd)

$$T_o = 42, \; C_p(T_o) = 4179$$

$$T_1 = 52, \; C_p(T_1) = 4186$$

$$T_2 = 82, \; C_p(T_2) = 4199$$

$$T_3 = 100, \; C_p(T_3) = 4217$$

$$L_0(T) = \prod_{\substack{j=0 \\ j \neq 0}}^{3} \frac{T - T_j}{T_0 - T_j} = \left(\frac{T - T_1}{T_0 - T_1}\right)\left(\frac{T - T_2}{T_0 - T_2}\right)\left(\frac{T - T_3}{T_0 - T_3}\right)$$

$$L_1(T) = \prod_{\substack{j=0 \\ j \neq 1}}^{3} \frac{T - T_j}{T_1 - T_j} = \left(\frac{T - T_0}{T_1 - T_0}\right)\left(\frac{T - T_2}{T_1 - T_2}\right)\left(\frac{T - T_3}{T_1 - T_3}\right)$$

$$L_2(T) = \prod_{\substack{j=0 \\ j \neq 2}}^{3} \frac{T - T_j}{T_2 - T_j} = \left(\frac{T - T_0}{T_2 - T_0}\right)\left(\frac{T - T_1}{T_2 - T_1}\right)\left(\frac{T - T_3}{T_2 - T_3}\right)$$

$$L_3(T) = \prod_{\substack{j=0 \\ j \neq 3}}^{3} \frac{T - T_j}{T_3 - T_j} = \left(\frac{T - T_0}{T_3 - T_0}\right)\left(\frac{T - T_1}{T_3 - T_1}\right)\left(\frac{T - T_2}{T_3 - T_2}\right)$$

Cubic Interpolation (contd)

$$C_p(T)=\left(\frac{T-T_1}{T_0-T_1}\right)\left(\frac{T-T_2}{T_0-T_2}\right)\left(\frac{T-T_3}{T_0-T_3}\right)C_p(T_0)+\left(\frac{T-T_0}{T_1-T_0}\right)\left(\frac{T-T_2}{T_1-T_2}\right)\left(\frac{T-T_3}{T_1-T_3}\right)C_p(T_1)$$

$$+\left(\frac{T-T_0}{T_2-T_0}\right)\left(\frac{T-T_1}{T_2-T_1}\right)\left(\frac{T-T_3}{T_2-T_3}\right)C_p(T_2)+\left(\frac{T-T_0}{T_3-T_0}\right)\left(\frac{T-T_1}{T_3-T_1}\right)\left(\frac{T-T_2}{T_3-T_2}\right)C_p(T_3)$$

$$C_p(61)=\frac{(61-52)(61-82)(61-100)}{(42-52)(42-82)(42-100)}(4179)+\frac{(61-42)(61-82)(61-100)}{(52-42)(52-82)(52-100)}(4186)$$

$$+\frac{(61-42)(61-52)(61-100)}{(82-42)(82-52)(82-100)}(4199)+\frac{(61-42)(61-52)(61-82)}{(100-42)(100-52)(100-82)}(4217)$$

$$=(-0.31772)(4179)+(1.0806)(4186)+(0.30875)(517.35)+(-0.071659)(4217)$$

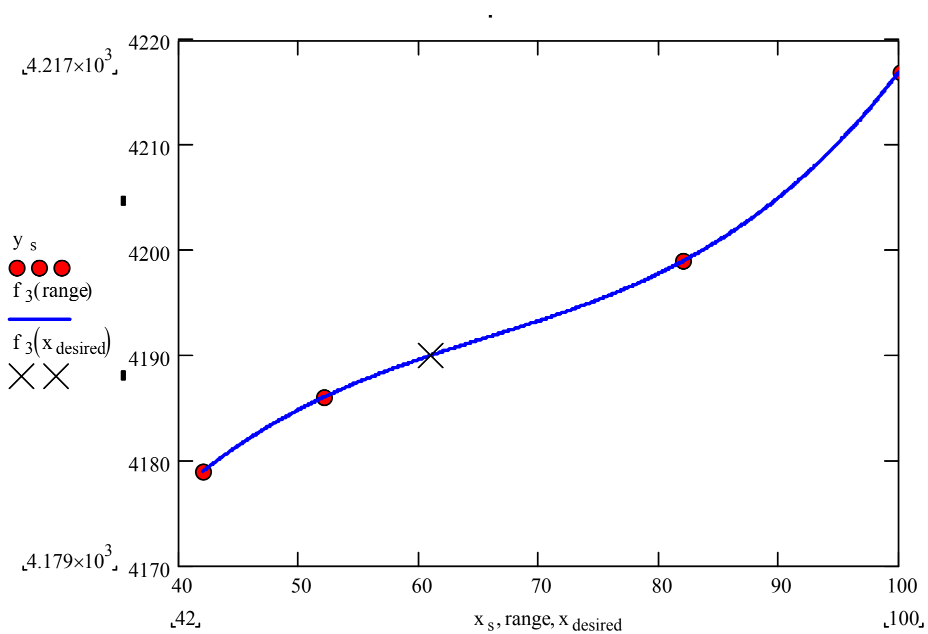
$$=4190.0\,\frac{J}{kg-^{\circ}C}$$

Cubic Interpolation (contd)

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a| = \left|\frac{4190.0 - 4191.2}{4190.0}\right| \times 100$$

$$= 0.028640\%$$



Comparison Table

Order of Polynomial	1	2	3
$C_p(T)$ $\frac{J}{kg-^{\circ}C}$	4189.9	4191.2	4190.0
Absolute Relative Approximate Error	----------	0.031017%	0.028640%