

## Chapter 05.02

# Direct Method of Interpolation

After reading this chapter, you should be able to:

1. apply the direct method of interpolation,
2. solve problems using the direct method of interpolation, and
3. use the direct method interpolants to find derivatives and integrals of discrete functions.

### What is interpolation?

Many a times, data is given only at discrete points such as  $(x_0, y_0)$ ,  $(x_1, y_1)$ , ...,  $(x_{n-1}, y_{n-1})$ ,  $(x_n, y_n)$ . So, how then does one find the value of  $y$  at any other value of  $x$ ? Well, a continuous function  $f(x)$  may be used to represent the  $n+1$  data values with  $f(x)$  passing through the  $n+1$  points (Figure 1). Then one can find the value of  $y$  at any other value of  $x$ . This is called *interpolation*.

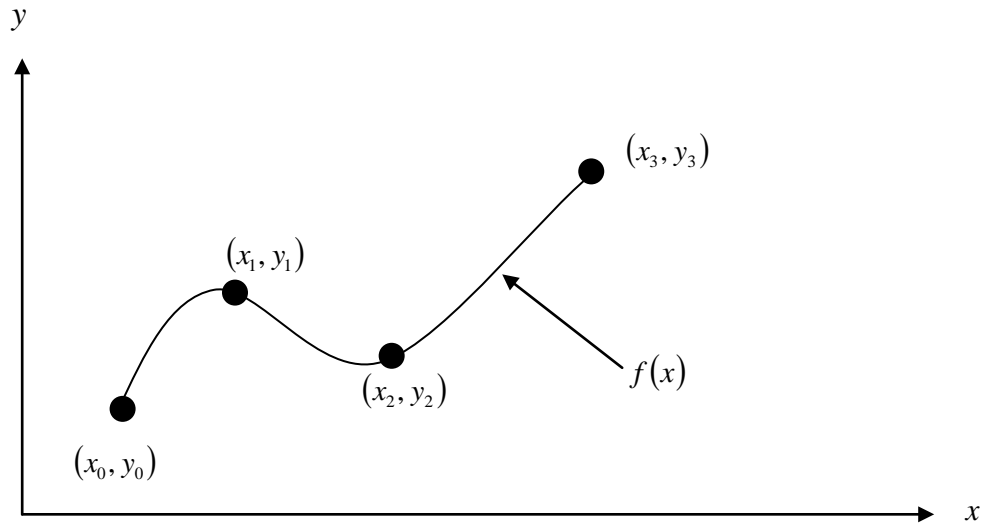
Of course, if  $x$  falls outside the range of  $x$  for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function  $f(x)$  should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

- (A) evaluate,
- (B) differentiate, and
- (C) integrate

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order  $n$  that passes through the  $n+1$  points. One of the methods of interpolation is called the direct method. Other methods include Newton's divided difference polynomial method and the Lagrangian interpolation method. We will discuss the direct method in this chapter.



**Figure 1** Interpolation of discrete data.

### Direct Method

The direct method of interpolation is based on the following premise. Given  $n + 1$  data points, fit a polynomial of order  $n$  as given below

$$y = a_0 + a_1x + \dots + a_nx^n \quad (1)$$

through the data, where  $a_0, a_1, \dots, a_n$  are  $n + 1$  real constants. Since  $n + 1$  values of  $y$  are given at  $n + 1$  values of  $x$ , one can write  $n + 1$  equations. Then the  $n + 1$  constants,  $a_0, a_1, \dots, a_n$  can be found by solving the  $n + 1$  simultaneous linear equations. To find the value of  $y$  at a given value of  $x$ , simply substitute the value of  $x$  in Equation 1.

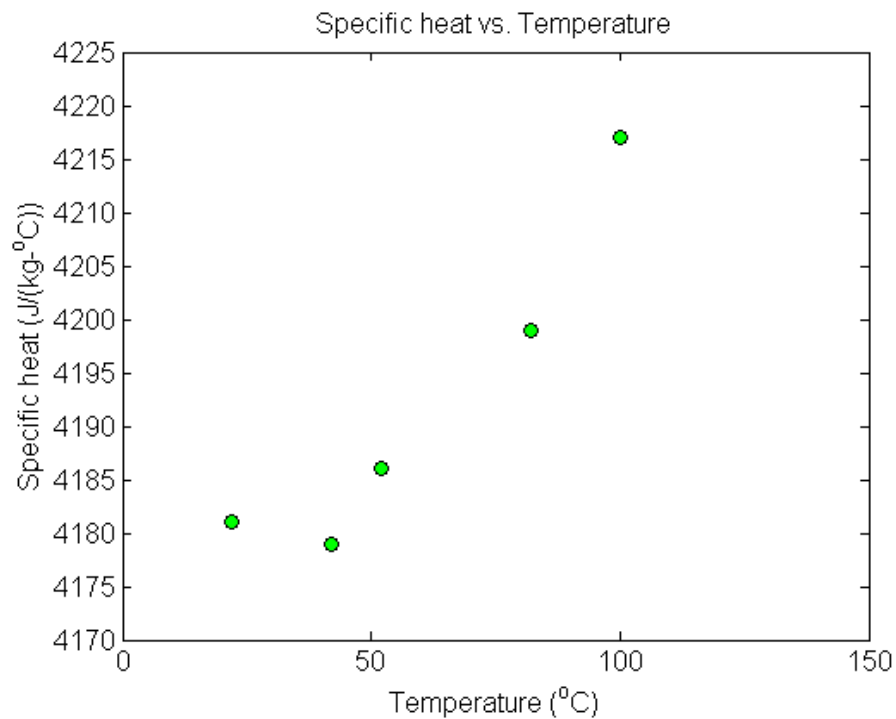
But, it is not necessary to use all the data points. How does one then choose the order of the polynomial and what data points to use? This concept and the direct method of interpolation are best illustrated using examples.

### Example 1

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at  $61^\circ\text{C}$ . The specific heat of water is given as a function of time in Table 1.

**Table 1** Specific heat of water as a function of temperature.

Temperature, $T$ ( $^{\circ}\text{C}$ )	Specific heat, $C_p$ $\left(\frac{\text{J}}{\text{kg} \cdot ^{\circ}\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

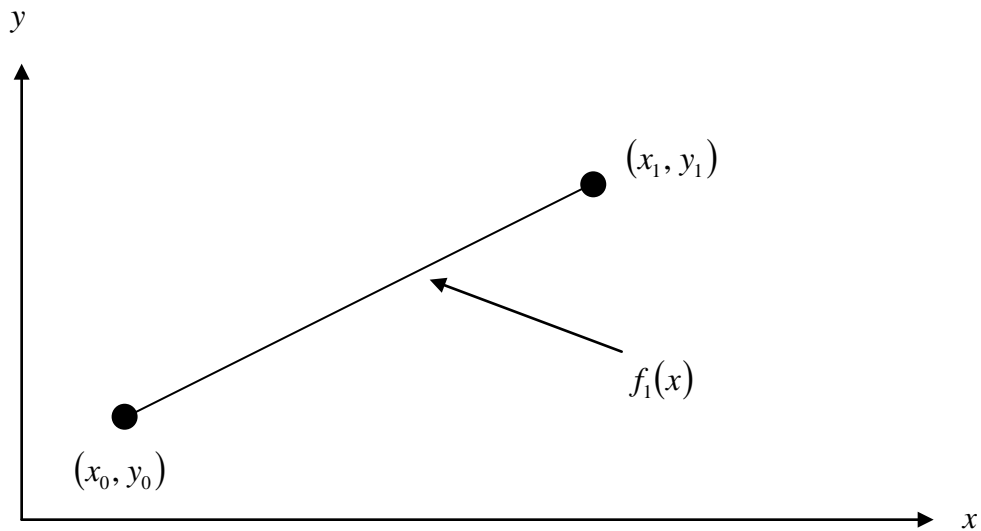
**Figure 2** Specific heat of water vs. temperature.

Determine the value of the specific heat at  $T = 61^{\circ}\text{C}$  using the direct method of interpolation and a first order polynomial.

**Solution**

For first order polynomial interpolation (also called linear interpolation), we choose the specific heat given by

$$C_p(T) = a_0 + a_1T$$



**Figure 3** Linear interpolation.

Since we want to find the specific heat at  $T = 61^\circ\text{C}$ , and we are using a first order polynomial, we need to choose the two data points that are closest to  $T = 61^\circ\text{C}$  that also bracket  $T = 61^\circ\text{C}$  to evaluate it. The two points are  $T_0 = 52$  and  $T_1 = 82$ .

Then

$$T_0 = 52, C_p(T_0) = 4186$$

$$T_1 = 82, C_p(T_1) = 4199$$

gives

$$C_p(52) = a_0 + a_1(52) = 4186$$

$$C_p(82) = a_0 + a_1(82) = 4199$$

Writing the equations in matrix form, we have

$$\begin{bmatrix} 1 & 52 \\ 1 & 82 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 4186 \\ 4199 \end{bmatrix}$$

Solving the above two equations gives

$$a_0 = 4163.5$$

$$a_1 = 0.43333$$

Hence

$$\begin{aligned} C_p(T) &= a_0 + a_1 T \\ &= 4163.5 + 0.43333T, \quad 52 \leq T \leq 82 \end{aligned}$$

At  $T = 61$ ,

$$\begin{aligned} C_p(61) &= 4163.5 + 0.43333(61) \\ &= 4189.9 \frac{\text{J}}{\text{kg} \cdot ^\circ\text{C}} \end{aligned}$$

**Example 2**

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at  $61^\circ\text{C}$ . The specific heat of water is given as a function of time in Table 2.

**Table 2** Specific heat of water as a function of temperature.

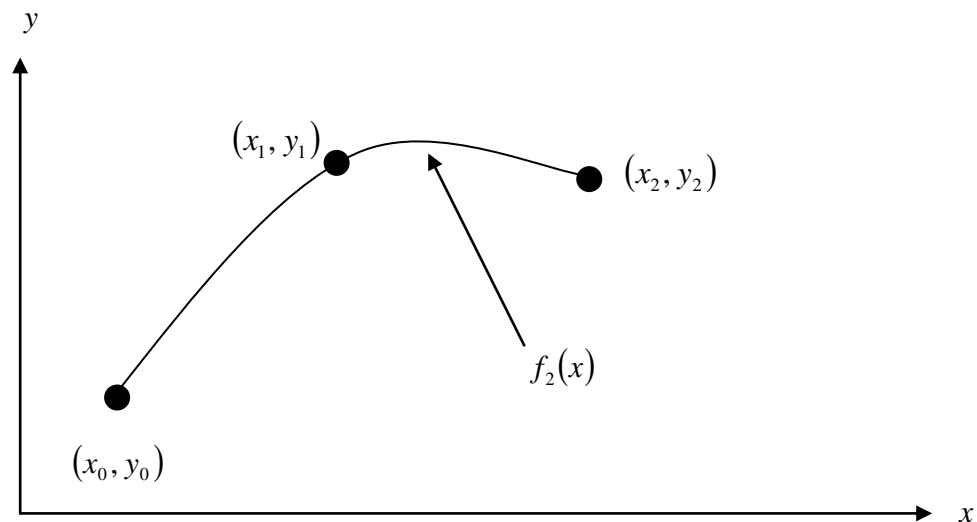
Temperature, $T$ ( $^\circ\text{C}$ )	Specific heat, $C_p$ $\left(\frac{\text{J}}{\text{kg} - ^\circ\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

Determine the value of the specific heat at  $T = 61^\circ\text{C}$  using the direct method of interpolation and a second order polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

**Solution**

For second order polynomial interpolation (also called quadratic interpolation), we choose the specific heat given by

$$C_p(T) = a_0 + a_1T + a_2T^2$$

**Figure 4** Quadratic interpolation.

Since we want to find the specific heat at  $T = 61^\circ\text{C}$ , and we are using a second order polynomial, we need to choose the three data points that are closest to  $T = 61^\circ\text{C}$  that also bracket  $T = 61^\circ\text{C}$  to evaluate it. The three points are  $T_0 = 42$ ,  $T_1 = 52$ , and  $T_2 = 82$ .

Then

$$T_0 = 42, C_p(T_0) = 4179$$

$$T_1 = 52, C_p(T_1) = 4186$$

$$T_2 = 82, C_p(T_2) = 4199$$

gives

$$C_p(42) = a_0 + a_1(42) + a_2(42)^2 = 4179$$

$$C_p(52) = a_0 + a_1(52) + a_2(52)^2 = 4186$$

$$C_p(82) = a_0 + a_1(82) + a_2(82)^2 = 4199$$

Writing the three equations in matrix form, we have

$$\begin{bmatrix} 1 & 42 & 1764 \\ 1 & 52 & 2704 \\ 1 & 82 & 6724 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4179 \\ 4186 \\ 4199 \end{bmatrix}$$

Solving the above three equations gives

$$a_0 = 4135.0$$

$$a_1 = 1.3267$$

$$a_2 = -6.6667 \times 10^{-3}$$

Hence

$$C_p(T) = 4135.0 + 1.3267T - 6.6667 \times 10^{-3}T^2, \quad 42 \leq T \leq 82$$

At  $T = 61$ ,

$$\begin{aligned} C_p(61) &= 4135.0 + 1.3267(61) - 6.6667 \times 10^{-3}(61)^2 \\ &= 4191.2 \frac{\text{J}}{\text{kg} \cdot ^\circ\text{C}} \end{aligned}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the first and second order polynomial is

$$\begin{aligned} |\epsilon_a| &= \left| \frac{4191.2 - 4189.9}{4191.2} \right| \times 100 \\ &= 0.030063\% \end{aligned}$$

### Example 3

To find how much heat is required to bring a kettle of water to its boiling point, you are asked to calculate the specific heat of water at  $61^\circ\text{C}$ . The specific heat of water is given as a function of time in Table 3.

**Table 3** Specific heat of water as a function of temperature.

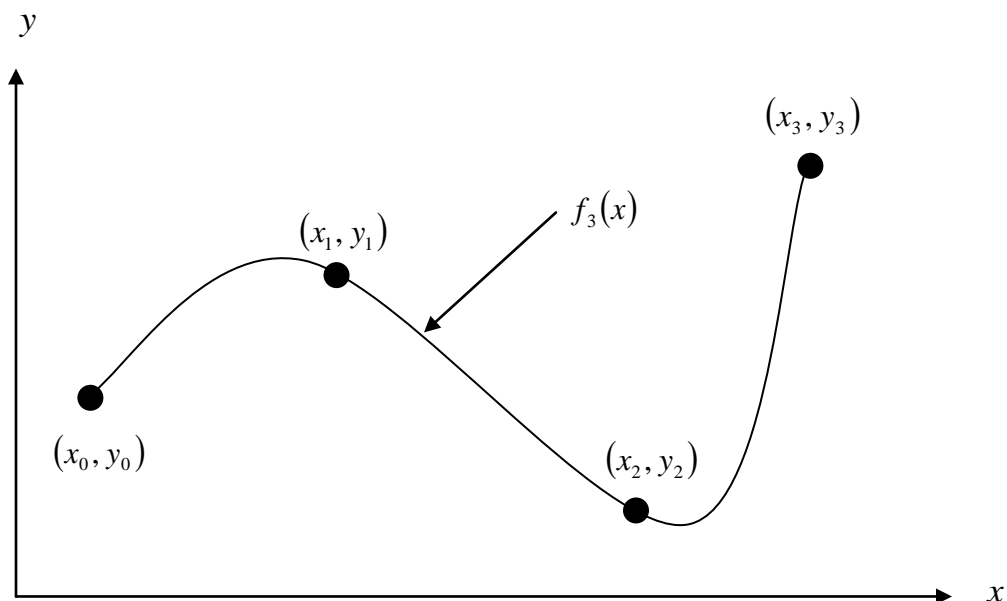
Temperature, $T$ (°C)	Specific heat, $C_p$ $\left(\frac{\text{J}}{\text{kg} \cdot ^\circ\text{C}}\right)$
22	4181
42	4179
52	4186
82	4199
100	4217

Determine the value of the specific heat at  $T = 61^\circ\text{C}$  using the direct method of interpolation and a third order polynomial. Find the absolute relative approximate error for the third order polynomial approximation.

**Solution**

For third order polynomial interpolation (also called cubic interpolation), we choose the specific heat given by

$$C_p(T) = a_0 + a_1T + a_2T^2 + a_3T^3$$

**Figure 5** Cubic interpolation.

Since we want to find the specific heat at  $T = 61^\circ\text{C}$ , and we are using a third order polynomial, we need to choose the four data points closest to  $T = 61^\circ\text{C}$  that also bracket

$T = 61^\circ\text{C}$  to evaluate it. The four points are  $T_0 = 42, T_1 = 52, T_2 = 82$  and  $T_3 = 100$ . (Choosing the four points as  $T_0 = 22, T_1 = 42, T_2 = 52$  and  $T_3 = 82$  is equally valid.)

Then

$$T_0 = 42, \quad C_p(T_0) = 4179$$

$$T_1 = 52, \quad C_p(T_1) = 4186$$

$$T_2 = 82, \quad C_p(T_2) = 4199$$

$$T_3 = 100, \quad C_p(T_3) = 4217$$

gives

$$C_p(42) = a_0 + a_1(42) + a_2(42)^2 + a_3(42)^3 = 4179$$

$$C_p(52) = a_0 + a_1(52) + a_2(52)^2 + a_3(52)^3 = 4186$$

$$C_p(82) = a_0 + a_1(82) + a_2(82)^2 + a_3(82)^3 = 4199$$

$$C_p(100) = a_0 + a_1(100) + a_2(100)^2 + a_3(100)^3 = 4217$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 42 & 1764 & 7.4088 \times 10^4 \\ 1 & 52 & 2704 & 1.4061 \times 10^5 \\ 1 & 82 & 6724 & 5.5137 \times 10^5 \\ 1 & 100 & 10000 & 10^6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4179 \\ 4186 \\ 4199 \\ 4217 \end{bmatrix}$$

Solving the above four equations gives

$$a_0 = 4078.0$$

$$a_1 = 4.4771$$

$$a_2 = -0.062720$$

$$a_3 = 3.1849 \times 10^{-4}$$

Hence

$$C_p(T) = a_0 + a_1T + a_2T^2 + a_3T^3$$

$$= 4078.0 + 4.4771T - 0.062720T^2 + 3.1849 \times 10^{-4}T^3, \quad 42 \leq T \leq 100$$

$$T(61) = 4078.0 + 4.4771(61) - 0.062720(61)^2 + 3.1849 \times 10^{-4}(61)^3$$

$$= 4190.0 \frac{\text{J}}{\text{kg} \cdot ^\circ\text{C}}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the second and third order polynomial is

$$\begin{aligned} |\epsilon_a| &= \left| \frac{4190.0 - 4191.2}{4190.0} \right| \times 100 \\ &= 0.027295\% \end{aligned}$$

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**INTERPOLATION**

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Topic	Direct Method of Interpolation
Summary	Textbook notes on the direct method of interpolation.
Major	Chemical Engineering
Authors	Autar Kaw, Peter Warr, Michael Keteltas
Date	August 27, 2009
Web Site	<a href="http://numericalmethods.eng.usf.edu">http://numericalmethods.eng.usf.edu</a>

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