Chapter 07.03
Simpson's 1/3 Rule of Integration

After reading this chapter, you should be able to

1. *derive the formula for Simpson's 1/3 rule of integration,*
2. *use Simpson's 1/3 rule it to solve integrals,*
3. *develop the formula for multiple-segment Simpson's 1/3 rule of integration,*
4. *use multiple-segment Simpson's 1/3 rule of integration to solve integrals, and*
5. *derive the true error formula for multiple-segment Simpson's 1/3 rule.*

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral. Here, we will discuss Simpson's 1/3 rule of integral approximation, which improves upon the accuracy of the trapezoidal rule.

Here, we will discuss the Simpson's 1/3 rule of approximating integrals of the form

$$I = \int_a^b f(x)dx$$

where

$f(x)$ is called the integrand,

$a =$ lower limit of integration

$b =$ upper limit of integration

Simpson's 1/3 Rule

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson's 1/3 rule is an extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

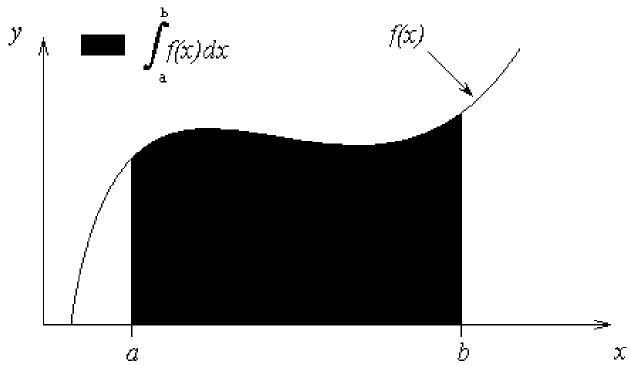

Figure 1 Integration of a function

Method 1:

Hence

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

where $f_2(x)$ is a second order polynomial given by

$$f_2(x) = a_0 + a_1 x + a_2 x^2 .$$

Choose

$$(a, f(a)), \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right), \text{ and } (b, f(b))$$

as the three points of the function to evaluate a_0, a_1 and a_2.

$$f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2$$

Solving the above three equations for unknowns, a_0, a_1 and a_2 give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Then

$$I \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}$$

Substituting values of a_0, a_1 and a_2 give

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since for Simpson 1/3 rule, the interval $[a,b]$ is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Hence the Simpson's 1/3 rule is given by

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since the above form has 1/3 in its formula, it is called Simpson's 1/3 rule.

<u>Method 2:</u>

Simpson's 1/3 rule can also be derived by approximating $f(x)$ by a second order polynomial using Newton's divided difference polynomial as

$$f_2(x) = b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right)$$

where

$$b_0 = f(a)$$

$$b_1 = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}$$

$$b_2 = \frac{\dfrac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} - \dfrac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}}{b-a}$$

Integrating Newton's divided difference polynomial gives us

$$\int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left[b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right)\right]dx$$

$$= \left[b_0 x + b_1\left(\frac{x^2}{2} - ax\right) + b_2\left(\frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2}\right)\right]_a^b$$

$$= b_0(b-a) + b_1\left(\frac{b^2-a^2}{2} - a(b-a)\right)$$

$$+ b_2\left(\frac{b^3-a^3}{3} - \frac{(3a+b)(b^2-a^2)}{4} + \frac{a(a+b)(b-a)}{2}\right)$$

Substituting values of b_0, b_1, and b_2 into this equation yields the same result as before

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

<u>Method 3:</u>

One could even use the Lagrange polynomial to derive Simpson's formula. Notice any method of three-point quadratic interpolation can be used to accomplish this task. In this case, the interpolating function becomes

$$f_2(x) = \frac{\left(x - \frac{a+b}{2}\right)(x-b)}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a) + \frac{(x-a)(x-b)}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) + \frac{(x-a)\left(x - \frac{a+b}{2}\right)}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b)$$

Integrating this function gets

$$\int_a^b f_2(x)dx = \left[\frac{\frac{x^3}{3} - \frac{(a+3b)x^2}{4} + \frac{b(a+b)x}{2}}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a) + \frac{\frac{x^3}{3} - \frac{(a+b)x^2}{2} + abx}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right) + \frac{\frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2}}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b) \right]_a^b$$

$$= \frac{\frac{b^3 - a^3}{3} - \frac{(a+3b)(b^2 - a^2)}{4} + \frac{b(a+b)(b-a)}{2}}{\left(a - \frac{a+b}{2}\right)(a-b)} f(a)$$

$$+ \frac{\frac{b^3 - a^3}{3} - \frac{(a+b)(b^2 - a^2)}{2} + ab(b-a)}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} f\left(\frac{a+b}{2}\right)$$

$$+ \frac{\frac{b^3 - a^3}{3} - \frac{(3a+b)(b^2 - a^2)}{4} + \frac{a(a+b)(b-a)}{2}}{(b-a)\left(b - \frac{a+b}{2}\right)} f(b)$$

Believe it or not, simplifying and factoring this large expression yields you the same result as before

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

<u>Method 4:</u>

Simpson's 1/3 rule can also be derived by the method of coefficients. Assume

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f\left(\frac{a+b}{2}\right) + c_3 f(b)$$

Let the right-hand side be an exact expression for the integrals $\int_a^b 1dx,\ \int_a^b xdx,$ and $\int_a^b x^2 dx$. This implies that the right hand side will be exact expressions for integrals of any linear combination of the three integrals for a general second order polynomial. Now

$$\int_a^b 1dx = b - a = c_1 + c_2 + c_3$$

$$\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1 a + c_2 \frac{a+b}{2} + c_3 b$$

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} = c_1 a^2 + c_2\left(\frac{a+b}{2}\right)^2 + c_3 b^2$$

Solving the above three equations for c_0, c_1 and c_2 give

$$c_1 = \frac{b-a}{6}$$

$$c_2 = \frac{2(b-a)}{3}$$
$$c_3 = \frac{b-a}{6}$$

This gives

$$\int_a^b f(x)dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)$$
$$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$
$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

The integral from the first method

$$\int_a^b f(x)dx \approx \int_a^b (a_0 + a_1 x + a_2 x^2)dx$$

can be viewed as the area under the second order polynomial, while the equation from Method 4

$$\int_a^b f(x)dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)$$

can be viewed as the sum of the areas of three rectangles.

Example 1

In an attempt to understand the mechanism of the depolarization process in a fuel cell, an electro-kinetic model for mixed oxygen-methanol current on platinum was developed in the laboratory at FAMU. A very simplified model of the reaction developed suggests a functional relation in an integral form. To find the time required for 50% of the oxygen to be consumed, the time, $T(s)$ is given by

$$T = -\int_{1.22\times10^{-6}}^{0.61\times10^{-6}} \left(\frac{6.73x + 4.3025\times10^{-7}}{2.316\times10^{-11}x}\right)dx$$

a) Use Simpson's 1/3 rule to find the time required for 50 % of the oxygen to be consumed.
b) Find the true error, E_t, for part (a).
c) Find the absolute relative true error, $|\in_t|$, for part (a).

Solution

a) $$T \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$a = 1.22\times10^{-6}$$

$$b = 0.61\times10^{-6}$$
$$\frac{a+b}{2} = 0.91500\times10^{-6}$$
$$f(x) = -\left[\frac{6.73x + 4.3025\times10^{-7}}{2.316\times10^{-11}x}\right]$$

$$T \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$
$$\approx \left(\frac{0.61\times10^{-6} - 1.22\times10^{-6}}{6}\right)\left[f(1.22\times10^{-6}) + 4f(0.915\times10^{-6}) + f(0.61\times10^{-6})\right]$$
$$\approx \left(\frac{-0.61\times10^{-6}}{6}\right)\left[-3.0581\times10^{11} + 4(-3.1089\times10^{11}) - 3.2104\times10^{11}\right]$$
$$\approx 190160 \text{ s}$$

Since

$$f(1.22\times10^{-6}) = -\left[\frac{6.73(1.22\times10^{-6}) + 4.3025\times10^{-7}}{2.316\times10^{-11}(1.22\times10^{-6})}\right]$$
$$= -3.0581\times10^{11}$$

$$f(0.61\times10^{-6}) = -\left[\frac{6.73(0.61\times10^{-6}) + 4.3025\times10^{-7}}{2.316\times10^{-11}(0.61\times10^{-6})}\right]$$
$$= -3.2104\times10^{11}$$

$$f(0.91500\times10^{-6}) = -\left[\frac{6.73(0.91500\times10^{-6}) + 4.3025\times10^{-7}}{2.316\times10^{-11}(0.91500\times10^{-6})}\right]$$
$$= -3.1089\times10^{11}$$

b) The exact value of the above integral is,

$$T = -\int_{1.22\times10^{-6}}^{0.61\times10^{-6}}\left(\frac{6.73x + 4.3025\times10^{-7}}{2.316\times10^{-11}x}\right)dx$$
$$= 1.90140\times10^{5}\text{ s}$$

so the true error is

$$E_t = \textit{True Value} - \textit{Approximate Value}$$
$$= 1.90140\times10^{5} - 190160$$
$$= -24.100$$

c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right|\times100\,\%$$

$$= \left| \frac{-24.020}{1.90140 \times 10^5} \right| \times 100\,\%$$
$$= 0.012675\,\%$$

Multiple-segment Simpson's 1/3 Rule

Just like in multiple-segment trapezoidal rule, one can subdivide the interval $[a,b]$ into n segments and apply Simpson's 1/3 rule repeatedly over every two segments. Note that n needs to be even. Divide interval $[a,b]$ into n equal segments, so that the segment width is given by

$$h = \frac{b-a}{n}.$$

Now

$$\int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx$$

where

$$x_0 = a$$
$$x_n = b$$

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots\ldots + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$

Apply Simpson's 1/3rd Rule over each interval,

$$\int_a^b f(x)dx \cong (x_2 - x_0)\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right] + (x_4 - x_2)\left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6}\right] + \ldots$$

$$+ (x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6}\right] + (x_n - x_{n-2})\left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}\right]$$

Since

$$x_i - x_{i-2} = 2h$$
$$i = 2,\ 4,\ \ldots,\ n$$

then

$$\int_a^b f(x)dx \cong 2h\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right] + 2h\left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6}\right] + \ldots$$
$$+ 2h\left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6}\right] + 2h\left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\{f(x_1) + f(x_3) + \ldots + f(x_{n-1})\} + 2\{f(x_2) + f(x_4) + \ldots + f(x_{n-2})\} + f(x_n)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n) \right]$$

$$\int_a^b f(x)dx \cong \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n) \right]$$

Example 2

In an attempt to understand the mechanism of the depolarization process in a fuel cell, an electro-kinetic model for mixed oxygen-methanol current on platinum was developed in the laboratory at FAMU. A very simplified model of the reaction developed suggests a functional relation in an integral form. To find the time required for 50% of the oxygen to be consumed, the time, $T(s)$ is given by

$$T = -\int_{1.22\times10^{-6}}^{0.61\times10^{-6}} \left(\frac{6.73x + 4.3025\times10^{-7}}{2.316\times10^{-11}x} \right) dx$$

a) Use four-Simpson's 1/3 Rule to find the time required for 50% of the oxygen to be consumed.
b) Find the true error, E_t, for part (a).
c) Find the absolute relative true error, $|\in_t|$, for part (a).

Solution

a) $$T \approx \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n) \right]$$

$$n = 4$$
$$a = 1.22\times10^{-6}$$
$$b = 0.61\times10^{-6}$$
$$h = \frac{b-a}{n}$$
$$= \frac{0.61\times10^{-6} - 1.22\times10^{-6}}{4}$$
$$= -0.15250\times10^{-6}$$
$$f(x) = -\left[\frac{6.73x + 4.3025\times10^{-7}}{2.316\times10^{-11}x} \right]$$

$$T \approx \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n)\right]$$

$$\approx \frac{0.61\times 10^{-6} - 1.22\times 10^{-6}}{3(4)}\left[f\left(1.22\times 10^{-6}\right) + 4\sum_{\substack{i=1 \\ i=odd}}^{3} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{2} f(x_i) + f\left(0.61\times 10^{-6}\right)\right]$$

$$\approx \frac{-0.61\times 10^{-6}}{12}\left[f\left(1.22\times 10^{-6}\right) + 4f(x_1) + 4f(x_3) + 2f(x_2) + f\left(0.61\times 10^{-6}\right)\right]$$

$$\approx \frac{-0.61\times 10^{-6}}{12}\left[\begin{matrix} f\left(1.22\times 10^{-6}\right) + 4f\left(1.0675\times 10^{-6}\right) + \\ 4f\left(0.76250\times 10^{-6}\right) + 2f\left(0.915\times 10^{-6}\right) + f\left(0.61\times 10^{-6}\right)\end{matrix}\right]$$

$$\approx \frac{-0.61\times 10^{-6}}{12}\left[\begin{matrix} -3.0582\times 10^{11} + 4\left(-3.0799\times 10^{11}\right) + \\ 4\left(-3.1495\times 10^{11}\right) + 2\left(-3.1089\times 10^{11}\right) - 3.2104\times 10^{11}\end{matrix}\right]$$

$$\approx 190140 \text{ s}$$

Since

$$f(x_0) = f\left(1.22\times 10^{-6}\right)$$
$$= -\left[\frac{6.73\left(1.22\times 10^{-6}\right) + 4.3025\times 10^{-7}}{2.316\times 10^{-11}\left(1.22\times 10^{-6}\right)}\right]$$
$$= -3.0581\times 10^{11}$$

$$f(x_1) = f\left(1.22\times 10^{-6} - 0.15250\times 10^{-6}\right)$$
$$= f\left(1.0675\times 10^{-6}\right)$$
$$= -\left[\frac{6.73\left(1.0675\times 10^{-6}\right) + 4.3025\times 10^{-7}}{2.316\times 10^{-11}\left(1.0675\times 10^{-6}\right)}\right]$$
$$= -3.0799\times 10^{11}$$

$$f(x_2) = f\left(1.0675\times 10^{-6} - 0.15250\times 10^{-6}\right)$$
$$= f\left(0.915\times 10^{-6}\right)$$
$$= -\left[\frac{6.73\left(0.915\times 10^{-6}\right) + 4.3025\times 10^{-7}}{2.316\times 10^{-11}\left(0.915\times 10^{-6}\right)}\right]$$
$$= -3.1089\times 10^{11}$$

$$f(x_3) = f\left(0.915\times 10^{-6} - 0.15250\times 10^{-6}\right)$$
$$= f\left(0.76250\times 10^{-6}\right)$$

$$= -\left[\frac{6.73(0.76250 \times 10^{-6}) + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11}(0.76250 \times 10^{-6})}\right]$$
$$= -3.1495 \times 10^{11}$$

$$f(x_4) = f(x_n)$$
$$= f(0.61 \times 10^{-6})$$
$$= -\left[\frac{6.73(0.61 \times 10^{-6}) + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11}(0.61 \times 10^{-6})}\right]$$
$$= -3.2104 \times 10^{11}$$

b) The exact value of the above integral is

$$T = -\int_{1.22\times10^{-6}}^{0.61\times10^{-6}} \left(\frac{6.73x + 4.3025 \times 10^{-7}}{2.316 \times 10^{-11}x}\right)dx$$
$$= 1.90140 \times 10^{5} \text{ s}$$

so the true error is

$$E_t = \textit{True Value} - \textit{Approximate Value}$$
$$= 1.90140 \times 10^{5} - 190140$$
$$= -1.9838$$

c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100\,\%$$
$$= \left|\frac{-1.9838}{1.90140 \times 10^{5}}\right| \times 100\,\%$$
$$= 0.0010434\,\%$$

Table 1 Values of Simpson’s 1/3 Rule for Example 2 with multiple segments.

n	Approximate Value	E_t	$\lvert\in_t\rvert$ %
2	190160	–24.100	0.012675
4	190140	–1.9838	0.0010434
6	190140	–0.42010	2.2094×10^{-4}
8	190140	–0.13655	7.1815×10^{-5}
10	190140	–0.056663	2.9802×10^{-5}

Error in Multiple-segment Simpson's 1/3 rule

The true error in a single application of Simpson's 1/3 Rule is given[1] by

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

In multiple-segment Simpson's 1/3 rule, the error is the sum of the errors in each application of Simpson's 1/3 rule. The error in the *n* segments Simpson's 1/3 Rule is given by

$$E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2$$

$$= -\frac{h^5}{90} f^{(4)}(\zeta_1)$$

$$E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4$$

$$= -\frac{h^5}{90} f^{(4)}(\zeta_2)$$

$$\vdots$$

$$E_i = -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i}$$

$$= -\frac{h^5}{90} f^{(4)}(\zeta_i)$$

$$\vdots$$

$$E_{\frac{n}{2}-1} = -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right), \quad x_{n-4} < \zeta_{\frac{n}{2}-1} < x_{n-2}$$

$$= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right)$$

$$E_{\frac{n}{2}} = -\frac{(x_n - x_{n-2})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}}\right), x_{n-2} < \zeta_{\frac{n}{2}} < x_n$$

Hence, the total error in the multiple-segment Simpson's 1/3 rule is

$$= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right)$$

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i$$

[1] The $f^{(4)}$ in the true error expression stands for the fourth derivative of the function *f(x)*.

$$= -\frac{h^5}{90}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^5}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^4}\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

The term $\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$ is an approximate average value of $f^{(4)}(x), a < x < b$. Hence

$$E_t = -\frac{(b-a)^5}{90n^4}\overline{f}^{(4)}$$

where

$$\overline{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

INTEGRATION	
Topic	Simpson's 1/3 rule
Summary	Textbook notes of Simpson's 1/3 rule
Major	Chemical Engineering
Authors	Autar Kaw, Michael Keteltas
Date	August 27, 2009
Web Site	http://numericalmethods.eng.usf.edu