Chapter 07.02
Trapezoidal Rule of Integration

After reading this chapter, you should be able to:

1. *derive the trapezoidal rule of integration,*
2. *use the trapezoidal rule of integration to solve problems,*
3. *derive the multiple-segment trapezoidal rule of integration,*
4. *use the multiple-segment trapezoidal rule of integration to solve problems, and*
5. *derive the formula for the true error in the multiple-segment trapezoidal rule of integration.*

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral.

Here, we will discuss the trapezoidal rule of approximating integrals of the form

$$I=\int_a^b f(x)dx$$

where

$f(x)$ is called the integrand,

$a =$ lower limit of integration

$b =$ upper limit of integration

What is the trapezoidal rule?

The trapezoidal rule is based on the Newton-Cotes formula that if one approximates the integrand by an n^{th} order polynomial, then the integral of the function is approximated by the integral of that n^{th} order polynomial. Integrating polynomials is simple and is based on the calculus formula.

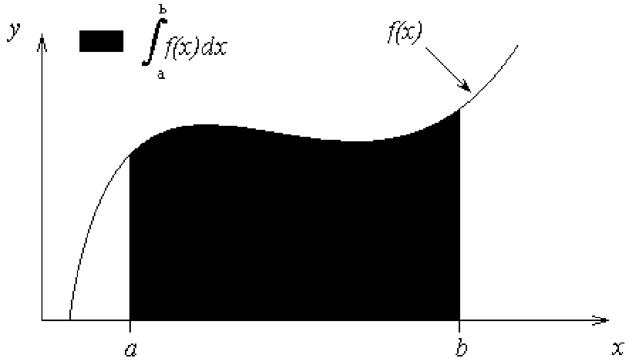

Figure 1 Integration of a function

$$\int_a^b x^n dx = \left(\frac{b^{n+1} - a^{n+1}}{n+1}\right),\ n \neq -1 \tag{1}$$

So if we want to approximate the integral

$$I = \int_a^b f(x)dx \tag{2}$$

to find the value of the above integral, one assumes

$$f(x) \approx f_n(x) \tag{3}$$

where

$$f_n(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n\ . \tag{4}$$

where $f_n(x)$ is a n^{th} order polynomial. The trapezoidal rule assumes $n = 1$, that is, approximating the integral by a linear polynomial (straight line),

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx$$

Derivation of the Trapezoidal Rule

<u>Method 1: Derived from Calculus</u>

$$\begin{aligned}\int_a^b f(x)dx &\approx \int_a^b f_1(x)dx \\ &= \int_a^b (a_0 + a_1 x)dx \\ &= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) \end{aligned} \tag{5}$$

But what is a_0 and a_1? Now if one chooses, $(a, f(a))$ and $(b, f(b))$ as the two points to approximate $f(x)$ by a straight line from a to b,

$$f(a) = f_1(a) = a_0 + a_1 a \tag{6}$$
$$f(b) = f_1(b) = a_0 + a_1 b \tag{7}$$

Solving the above two equations for a_1 and a_0,

$$a_1 = \frac{f(b) - f(a)}{b-a}$$
$$a_0 = \frac{f(a)b - f(b)a}{b-a} \tag{8a}$$

Hence from Equation (5),

$$\int_a^b f(x)dx \approx \frac{f(a)b - f(b)a}{b-a}(b-a) + \frac{f(b)-f(a)}{b-a}\frac{b^2-a^2}{2} \tag{8b}$$
$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right] \tag{9}$$

<u>Method 2: Also Derived from Calculus</u>

$f_1(x)$ can also be approximated by using Newton's divided difference polynomial as

$$f_1(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \tag{10}$$

Hence

$$\begin{aligned}
\int_a^b f(x)dx &\approx \int_a^b f_1(x)dx \\
&= \int_a^b \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]dx \\
&= \left[f(a)x + \frac{f(b)-f(a)}{b-a}\left(\frac{x^2}{2} - ax\right)\right]_a^b \\
&= f(a)b - f(a)a + \left(\frac{f(b)-f(a)}{b-a}\right)\left(\frac{b^2}{2} - ab - \frac{a^2}{2} + a^2\right) \\
&= f(a)b - f(a)a + \left(\frac{f(b)-f(a)}{b-a}\right)\left(\frac{b^2}{2} - ab + \frac{a^2}{2}\right) \\
&= f(a)b - f(a)a + \left(\frac{f(b)-f(a)}{b-a}\right)\frac{1}{2}(b-a)^2 \\
&= f(a)b - f(a)a + \frac{1}{2}(f(b) - f(a))(b-a)
\end{aligned}$$

$$= f(a)b - f(a)a + \frac{1}{2}f(b)b - \frac{1}{2}f(b)a - \frac{1}{2}f(a)b + \frac{1}{2}f(a)a$$

$$= \frac{1}{2}f(a)b - \frac{1}{2}f(a)a + \frac{1}{2}f(b)b - \frac{1}{2}f(b)a$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right] \tag{11}$$

This gives the same result as Equation (10) because they are just different forms of writing the same polynomial.

Method 3: Derived from Geometry

The trapezoidal rule can also be derived from geometry. Look at Figure 2. The area under the curve $f_1(x)$ is the area of a trapezoid. The integral

$$\int_a^b f(x)dx \approx \text{Area of trapezoid}$$

$$= \frac{1}{2}(\text{Sum of length of parallel sides})(\text{Perpendicular distance between parallel sides})$$

$$= \frac{1}{2}\left(f(b)+f(a)\right)(b-a)$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right] \tag{12}$$

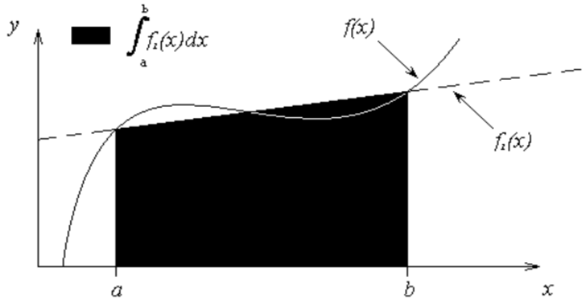

Figure 2 Geometric representation of trapezoidal rule.

Method 4: Derived from Method of Coefficients

The trapezoidal rule can also be derived by the method of coefficients. The formula

$$\int_a^b f(x)dx \approx \frac{b-a}{2}f(a) + \frac{b-a}{2}f(b) \tag{13}$$

$$= \sum_{i=1}^{2} c_i f(x_i)$$

where

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2}$$

$$x_1 = a$$

$$x_2 = b$$

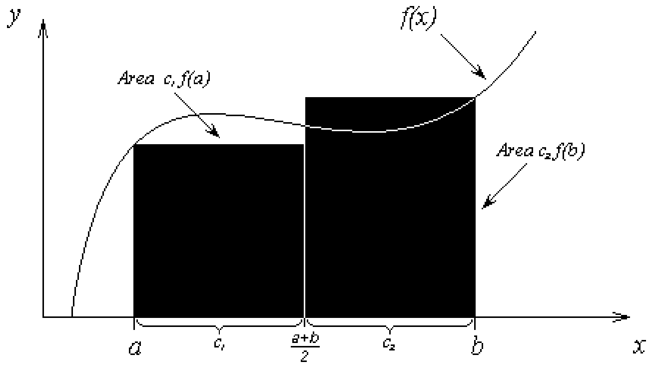

Figure 3 Area by method of coefficients.

The interpretation is that $f(x)$ is evaluated at points a and b, and each function evaluation is given a weight of $\frac{b-a}{2}$. Geometrically, Equation (12) is looked at as the area of a trapezoid, while Equation (13) is viewed as the sum of the area of two rectangles, as shown in Figure 3. How can one derive the trapezoidal rule by the method of coefficients?

Assume

$$\int_a^b f(x)dx = c_1 f(a) + c_2 f(b) \tag{14}$$

Let the right hand side be an exact expression for integrals of $\int_a^b 1dx$ and $\int_a^b xdx$, that is, the formula will then also be exact for linear combinations of $f(x)=1$ and $f(x)=x$, that is, for $f(x) = a_0(1) + a_1(x)$.

$$\int_a^b 1dx = b - a = c_1 + c_2 \tag{15}$$

$$\int_a^b xdx = \frac{b^2-a^2}{2} = c_1 a + c_2 b \tag{16}$$

Solving the above two equations gives

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2} \tag{17}$$

Hence

$$\int_a^b f(x)dx \approx \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b) \tag{18}$$

Method 5: Another approach on the Method of Coefficients

The trapezoidal rule can also be derived by the method of coefficients by another approach

$$\int_a^b f(x)dx \approx \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$

Assume

$$\int_a^b f(x)dx = c_1 f(a) + c_2 f(b) \tag{19}$$

Let the right hand side be exact for integrals of the form

$$\int_a^b (a_0 + a_1 x)dx$$

So

$$\int_a^b (a_0 + a_1 x)dx = \left(a_0 x + a_1 \frac{x^2}{2} \right)_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) \tag{20}$$

But we want

$$\int_a^b (a_0 + a_1 x)dx = c_1 f(a) + c_2 f(b) \tag{21}$$

to give the same result as Equation (20) for $f(x) = a_0 + a_1 x$.

$$\int_a^b (a_0 + a_1 x)dx = c_1(a_0 + a_1 a) + c_2(a_0 + a_1 b)$$

$$= a_0(c_1 + c_2) + a_1(c_1 a + c_2 b) \tag{22}$$

Hence from Equations (20) and (22),

$$a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) = a_0(c_1 + c_2) + a_1(c_1 a + c_2 b)$$

Since a_0 and a_1 are arbitrary for a general straight line

$$c_1 + c_2 = b - a$$

$$c_1 a + c_2 b = \frac{b^2 - a^2}{2} \tag{23}$$

Again, solving the above two equations (23) gives

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2} \tag{24}$$

Therefore

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b) \tag{25}$$

Example 1

In an attempt to understand the mechanism of the depolarization process in a fuel cell, an electro-kinetic model for mixed oxygen-methanol current on platinum was developed in the laboratory at FAMU. A very simplified model of the reaction developed suggests a functional relation in an integral form. To find the time required for 50% of the oxygen to be consumed, the time, $T(s)$ is given by

$$T = -\int_{1.22\times10^{-6}}^{0.61\times10^{-6}} \left(\frac{6.73x + 4.3025\times10^{-7}}{2.316\times10^{-11} x} \right) dx$$

a) Use single segment Trapezoidal rule to find the time required for 50% of the oxygen to be consumed.
b) Find the true error, E_t, for part (a).
c) Find the absolute relative true error, $|\in_t|$, for part (a).

Solution

a) $I \approx (b-a)\left[\frac{f(a) + f(b)}{2}\right]$, where

$$a = 1.22\times10^{-6}$$

$$b = 0.61\times10^{-6}$$

$$f(x) = -\left[\frac{6.73x + 4.3025\times10^{-7}}{2.316\times10^{-11} x}\right]$$

$$f(1.22\times10^{-6}) = -\left[\frac{6.73(1.22\times10^{-6}) + 4.3025\times10^{-7}}{2.316\times10^{-11}(1.22\times10^{-6})}\right] = -3.0581\times10^{11}$$

$$f(0.61\times10^{-6}) = -\left[\frac{6.73(0.61\times10^{-6}) + 4.3025\times10^{-7}}{2.316\times10^{-11}(0.61\times10^{-6})}\right] = -3.2104\times10^{11}$$

$$I = (0.61\times10^{-6} - 1.22\times10^{-6})\left[\frac{-3.0582\times10^{11} + (-3.2104\times10^{11})}{2}\right]$$

$$= 1.9119\times10^{5}\,\text{s}$$

b) The exact value of the above integral is,

$$T = -\int_{1.22\times10^{-6}}^{0.61\times10^{-6}} \left(\frac{6.73x + 4.3025\times10^{-7}}{2.316\times10^{-11}x} \right) dx$$

$$= 1.9014\times10^{5}\,\text{s}$$

so the true error is

$$E_t = \textit{True Value} - \textit{Approximate Value}$$

$$= 1.9014\times10^{5} - 1.9119\times10^{5}$$

$$= -1056.2$$

c) The absolute relative true error, $\left|\in_t\right|$, would then be

$$\left|\in_t\right| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{-1056.2}{1.9014\times10^{5}}\right| \times 100$$

$$= 0.55549\,\%$$

Multiple-Segment Trapezoidal Rule

In Example 1, the true error using a single segment trapezoidal rule was large. We can divide the interval $[8,30]$ into $[8,19]$ and $[19,30]$ intervals and apply the trapezoidal rule over each segment.

$$f(t) = 2000\ln\left(\frac{140000}{140000 - 2100t}\right) - 9.8t$$

$$\int_8^{30} f(t)dt = \int_8^{19} f(t)dt + \int_{19}^{30} f(t)dt$$

$$\approx (19-8)\left[\frac{f(8)+f(19)}{2}\right] + (30-19)\left[\frac{f(19)+f(30)}{2}\right]$$

$$f(8) = 177.27 \text{ m/s}$$

$$f(19) = 2000\ln\left(\frac{140000}{140000 - 2100(19)}\right) - 9.8(19) = 484.75 \text{ m/s}$$

$$f(30) = 901.67 \text{ m/s}$$

Hence

$$\int_8^{30} f(t)dt \approx (19-8)\left[\frac{177.27+484.75}{2}\right] + (30-19)\left[\frac{484.75+901.67}{2}\right]$$

$$= 11266 \text{ m}$$

The true error, E_t is

$$E_t = 11061 - 11266$$

$$= -205\,\text{m}$$

The true error now is reduced from 807 m to 205 m. Extending this procedure to dividing $[a,b]$ into n equal segments and applying the trapezoidal rule over each segment, the sum of the results obtained for each segment is the approximate value of the integral.

Divide $(b-a)$ into n equal segments as shown in Figure 4. Then the width of each segment is

$$h=\frac{b-a}{n} \qquad (26)$$

The integral I can be broken into h integrals as

$$I=\int_a^b f(x)dx$$

$$=\int_a^{a+h} f(x)dx+\int_{a+h}^{a+2h} f(x)dx+\ldots+\int_{a+(n-2)h}^{a+(n-1)h} f(x)dx+\int_{a+(n-1)h}^{b} f(x)dx \qquad (27)$$

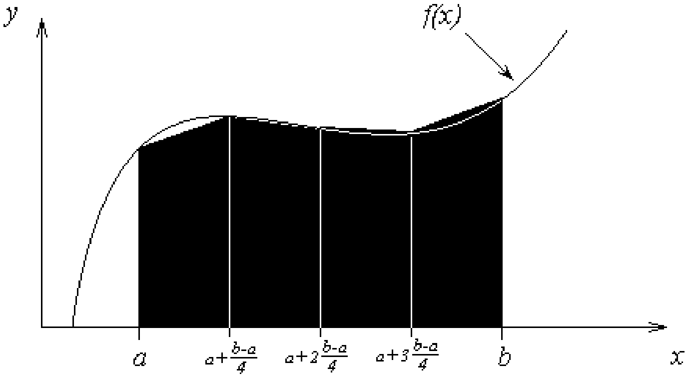

Figure 4 Multiple ($n=4$) segment trapezoidal rule

Applying trapezoidal rule Equation (27) on each segment gives

$$\int_a^b f(x)dx=\left[(a+h)-a\right]\left[\frac{f(a)+f(a+h)}{2}\right]$$

$$+\left[(a+2h)-(a+h)\right]\left[\frac{f(a+h)+f(a+2h)}{2}\right]$$

$$+\ldots\ldots\ldots\ldots\ldots\quad+\left[\left(a+(n-1)h\right)-\left(a+(n-2)h\right)\right]\left[\frac{f(a+(n-2)h)+f(a+(n-1)h)}{2}\right]$$

$$+\left[b-\left(a+(n-1)h\right)\right]\left[\frac{f(a+(n-1)h)+f(b)}{2}\right]$$

$$=h\left[\frac{f(a)+f(a+h)}{2}\right]+h\left[\frac{f(a+h)+f(a+2h)}{2}\right]+\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+h\left[\frac{f(a+(n-2)h)+f(a+(n-1)h)}{2}\right]+h\left[\frac{f(a+(n-1)h)+f(b)}{2}\right]$$

$$=h\left[\frac{f(a)+2f(a+h)+2f(a+2h)+\ldots+2f(a+(n-1)h)+f(b)}{2}\right]$$

$$= \frac{h}{2}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right]$$

$$= \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right] \tag{28}$$

Example 2

In an attempt to understand the mechanism of the depolarization process in a fuel cell, an electro-kinetic model for mixed oxygen-methanol current on platinum was developed in the laboratory at FAMU. A very simplified model of the reaction developed suggests a functional relation in an integral form. To find the time required for 50% of the oxygen to be consumed, the time, $T(s)$ is given by

$$T = -\int_{1.22\times10^{-6}}^{0.61\times10^{-6}}\left(\frac{6.73x + 4.3025\times10^{-7}}{2.316\times10^{-11}x}\right)dx$$

a) Use two- segment Trapezoidal rule to find the time required for 50% of the oxygen to be consumed.
b) Find the true error, E_t, for part (a).
c) Find the absolute relative true error, $|\in_t|$, for part (a).

Solution

a) $$I = \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right]$$

$n = 2$

$a = 1.22\times10^{-6}$

$b = 0.61\times10^{-6}$

$$f(x) = -\left[\frac{6.73x + 4.3025\times10^{-7}}{2.316\times10^{-11}x}\right]$$

$$h = \frac{b-a}{n}$$

$$= \frac{0.61\times10^{-6} - 1.22\times10^{-6}}{2}$$

$$= -0.30500\times10^{-6}$$

$$f(x_0) = f\left(1.22\times10^{-6}\right)$$

$$f\left(1.22\times10^{-6}\right) = -\left[\frac{6.73\left(1.22\times10^{-6}\right) + 4.3025\times10^{-7}}{2.316\times10^{-11}\left(1.22\times10^{-6}\right)}\right] = -3.0581\times10^{11}$$

$$f(x_1) = f\left(1.22\times10^{-6} - 0.30500\times10^{-6}\right)$$

$$= f\left(0.91500\times10^{-6}\right)$$

$$f(0.91500) = -\left[\frac{6.73(0.915\times10^{-6}) + 4.3025\times10^{-7}}{2.316\times10^{-11}(0.915\times10^{-6})}\right] = -3.1089\times10^{11}$$

$$f(x_2) = f(x_n) = f(0.61\times10^{-6})$$

$$f(0.61\times10^{-6}) = -\left[\frac{6.73(0.61\times10^{-6}) + 4.3025\times10^{-7}}{2.316\times10^{-11}(0.61\times10^{-6})}\right] = -3.2104\times10^{11}$$

$$I = \frac{0.61\times10^{-6} - 1.22\times10^{-6}}{2(2)}\left[f(1.22\times10^{-6}) + 2\left\{\sum_{i=1}^{2-1} f(a+ih)\right\} + f(0.61\times10^{-6})\right]$$

$$= \frac{-0.61\times10^{-6}}{4}\left[f(1.22\times10^{-6}) + 2f(0.915\times10^{-6}) + f(0.61\times10^{-6})\right]$$

$$= \frac{-0.61\times10^{-6}}{4}\left[-3.0581\times10^{11} + 2(-3.1089\times10^{11}) - 3.2104\times10^{11}\right]$$

$$= 1.9042\times10^{5}\,\text{s}$$

b) The exact value of the above integral is,

$$T = -\int_{1.22\times10^{-6}}^{0.61\times10^{-6}}\left(\frac{6.73x + 4.3025\times10^{-7}}{2.316\times10^{-11}x}\right)dx$$

$$= 1.90140\times10^{5}\,\text{s}$$

so the true error is

$$E_t = \textit{True Value} - \textit{Approximate Value}$$

$$= 1.90140\times10^{5} - 1.9042\times10^{5}$$

$$= -282.12$$

c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right|\times100$$

$$= \left|\frac{-282.12}{1.9014\times10^{5}}\right|\times100$$

$$= 0.14838\,\%$$

Table 1 Values obtained using multiple-segment Trapezoidal rule for

$$T = -\int_{1.22\times10^{-6}}^{0.61\times10^{-6}} \left(\frac{6.73x + 4.3025\times10^{-7}}{2.316\times10^{-11}x} \right) dx$$

n	Value	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
1	191190	-1056.2	0.55549	---
2	190420	-282.12	0.14838	0.40711
3	190260	-127.31	0.066956	0.081424
4	190210	-72.017	0.037877	0.029079
5	190180	-46.216	0.024307	0.013570
6	190170	-32.142	0.016905	0.0074020
7	190160	-23.636	0.012431	0.0044740
8	190150	-18.107	0.0095231	0.0029079

Error in Multiple-segment Trapezoidal Rule

The true error for a single segment Trapezoidal rule is given by

$$E_t = -\frac{(b-a)^3}{12} f''(\zeta), \quad a < \zeta < b$$

Where ζ is some point in $[a,b]$.

What is the error then in the multiple-segment trapezoidal rule? It will be simply the sum of the errors from each segment, where the error in each segment is that of the single segment trapezoidal rule. The error in each segment is

$$E_1 = -\frac{[(a+h)-a]^3}{12} f''(\zeta_1), \quad a < \zeta_1 < a+h$$

$$= -\frac{h^3}{12} f''(\zeta_1)$$

$$E_2 = -\frac{[(a+2h)-(a+h)]^3}{12} f''(\zeta_2), \quad a+h < \zeta_2 < a+2h$$

$$= -\frac{h^3}{12} f''(\zeta_2)$$

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$$E_i = -\frac{[(a+ih)-(a+(i-1)h)]^3}{12} f''(\zeta_i), \quad a+(i-1)h < \zeta_i < a+ih$$

$$= -\frac{h^3}{12} f''(\zeta_i)$$

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$$E_{n-1} = -\frac{\left[\{a+(n-1)h\} - \{a+(n-2)h\}\right]^3}{12} f''(\zeta_{n-1}), \quad a+(n-2)h < \zeta_{n-1} < a+(n-1)h$$

$$= -\frac{h^3}{12} f''(\zeta_{n-1})$$

$$E_n = -\frac{\left[b - \{a+(n-1)h\}\right]^3}{12} f''(\zeta_n), \quad a+(n-1)h < \zeta_n < b$$

$$= -\frac{h^3}{12} f''(\zeta_n)$$

Hence the total error in the multiple-segment trapezoidal rule is

$$E_t = \sum_{i=1}^{n} E_i$$

$$= -\frac{h^3}{12} \sum_{i=1}^{n} f''(\zeta_i)$$

$$= -\frac{(b-a)^3}{12n^3} \sum_{i=1}^{n} f''(\zeta_i)$$

$$= -\frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

The term $\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$ is an approximate average value of the second derivative $f''(x),\ a < x < b$.

Hence

$$E_t = -\frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

INTEGRATION	
Topic	Trapezoidal Rule
Summary	These are textbook notes of trapezoidal rule of integration
Major	Chemical Engineering
Authors	Autar Kaw, Michael Keteltas
Date	August 27, 2009
Web Site	http://numericalmethods.eng.usf.edu