Ordinary Differential Equations

Topic: Euler Method

Major: Chemical Engineering

Authors: Autar Kaw, Charlie Barker

Euler's Method

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

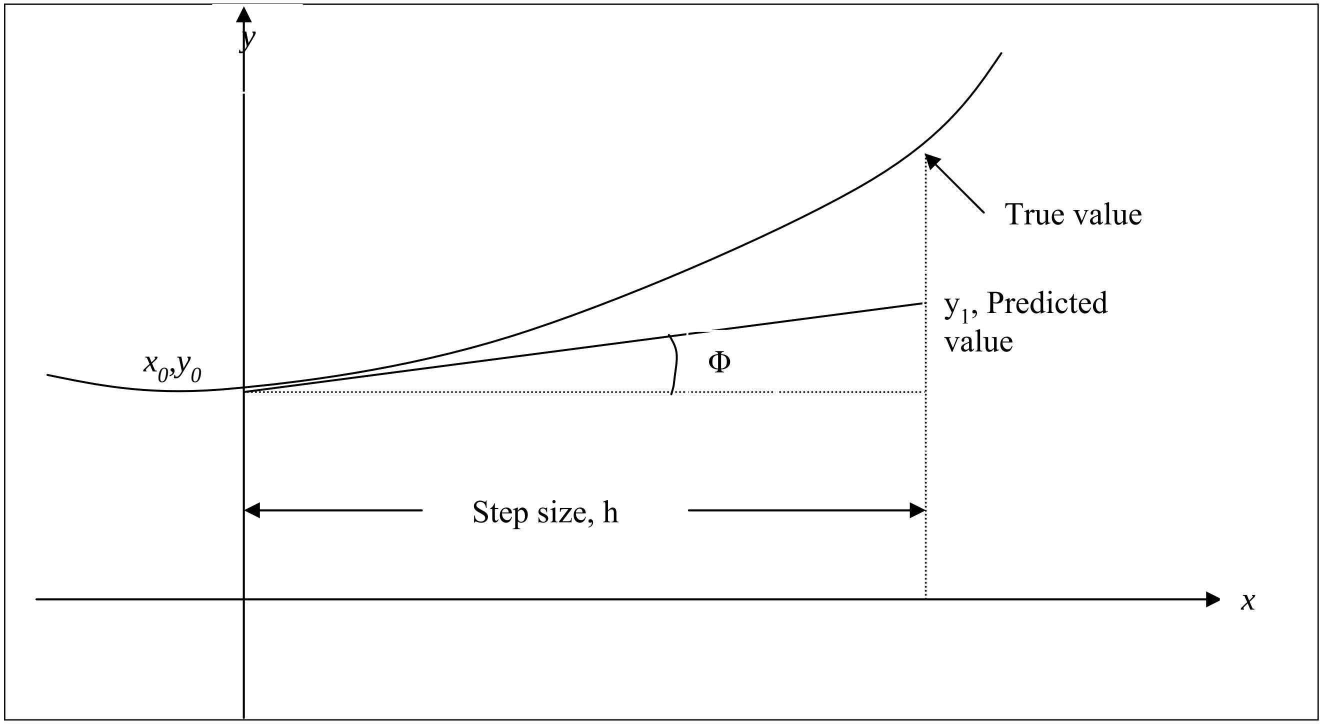

Figure 1. Graphical interpretation of the first step of Euler's method

Euler's Method

$$y_{i+1} = y_i + f\left(x_i, y_i\right)h$$

$$h = x_{i+1} - x_i$$

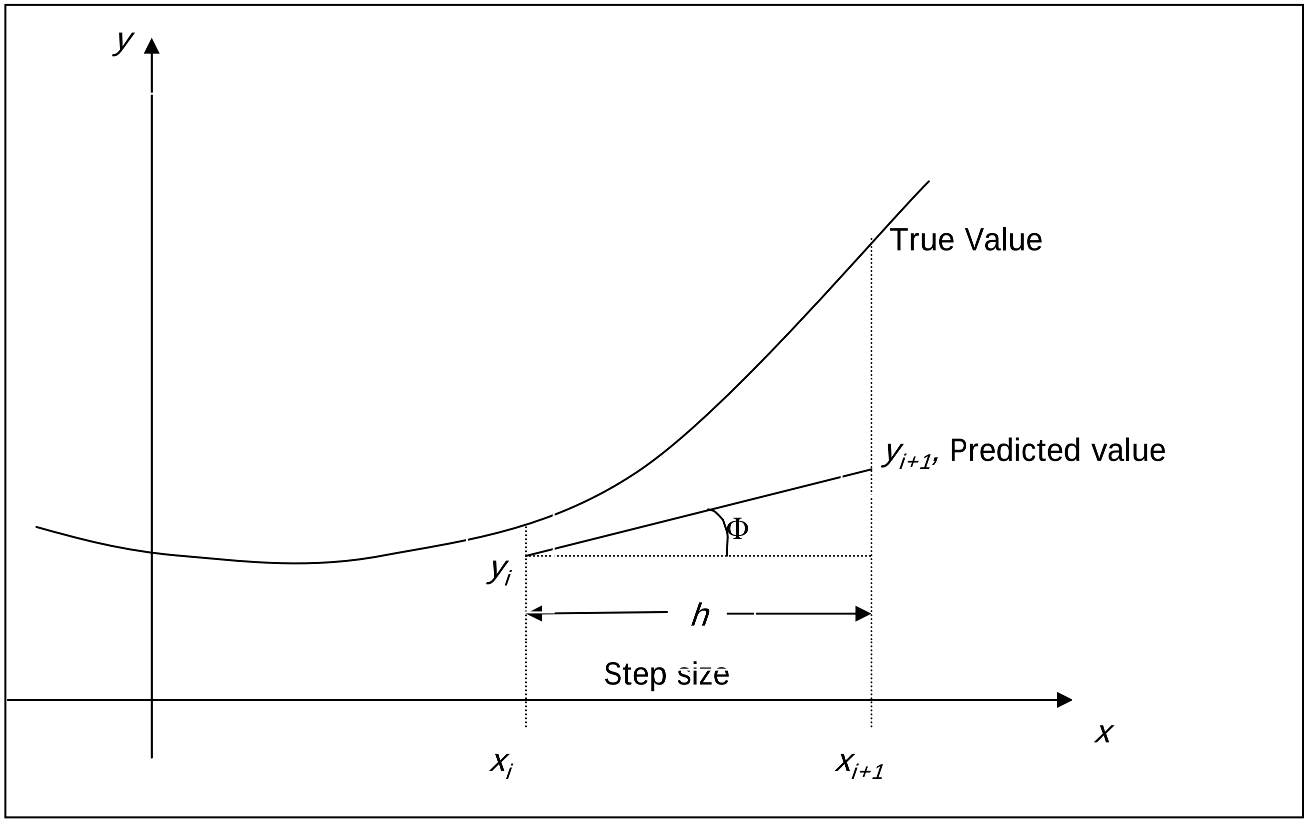

Figure 2. General graphical interpretation of Euler's method

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

The concentration of salt, x in a home made soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5x$$

At the initial time, *t = 0*, the salt concentration in the tank is *50g/L.* Using Euler's method and a step size of *h=1.5 min*, what is the salt concentration after *3* minutes.

$$\frac{dx}{dt} = 37.5 - 3.5x$$

$$f(t,x) = 37.5 - 3.5x$$

$$x_{i+1} = x_i + f(t_i, x_i)h$$

Solution

Step 1:

$$x_1 = x_0 + f(t_0, x_0)h$$
$$= 50 + f(0,50)1.5$$
$$= 50 + (37.5 - 3.5(50))1.5$$
$$= 50 + (-137.50)1.5$$
$$= -156.25 g/L$$

x_1 is the approximate concentration of salt at

$$t = t_1 = t_0 + h = 0 + 1.5 = 1.5 \min$$

$$x_1 = x(1.5) \cong -156.25 g/L$$

Solution Cont

Step 2:

$$x_2 = x_1 + f(t_1, x_1)h$$

$$= -156.25 + f(1.5, -156.25)1.5$$

$$= -156.25 + (37.5 - 3.5(-156.25))1.5$$

$$= -156.25 + (584.38)1.5$$

$$= 720.31 g/L$$

x_2 is the approximate concentration of salt at

$$t = t_2 = t_1 + h = 1.5 + 1.5 = 3 \min$$

$$x_2 = x(3) \cong 720.31 g/L$$

Solution Cont

The exact solution of the ordinary differential equation is given by

$$x(t) = 10.714 + 39.286e^{-3.5x}$$

The solution to this nonlinear equation at t=3 minutes is

$$x(3) = 10.715$$

Comparison of Exact and Numerical Solutions

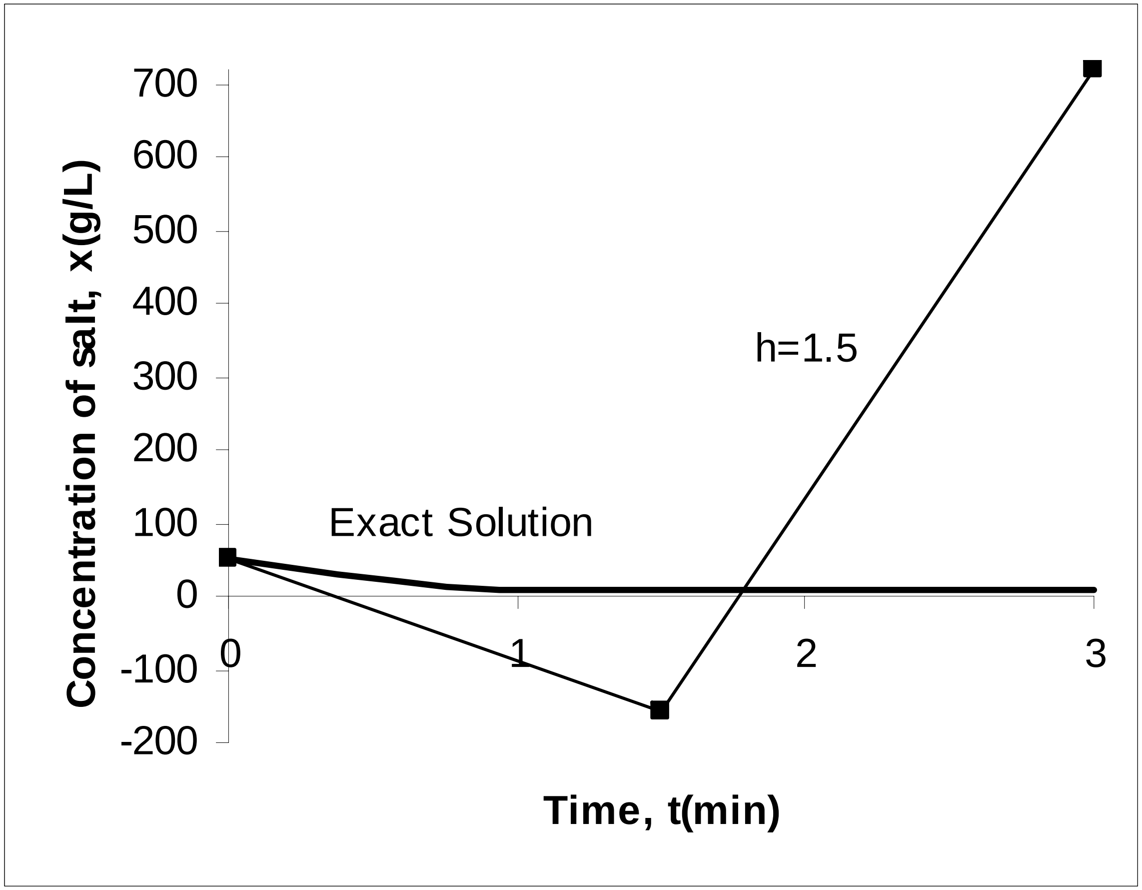

Figure 3. Comparing exact and Euler's method

Effect of step size

Table 1. Concentration of salt at 3 minutes as a function of step size, h

Step h	$x(3)$	E_t	$\lvert \in_t \rvert$ %
3	-362.50	373.22	3483.0
1.5	720.31	-709.60	6622.2
0.75	284.65	-273.93	2556.5
0.375	10.718	-0.0024912	0.023249
0.1875	10.714	0.0010803	0.010082

Comparison with exact results

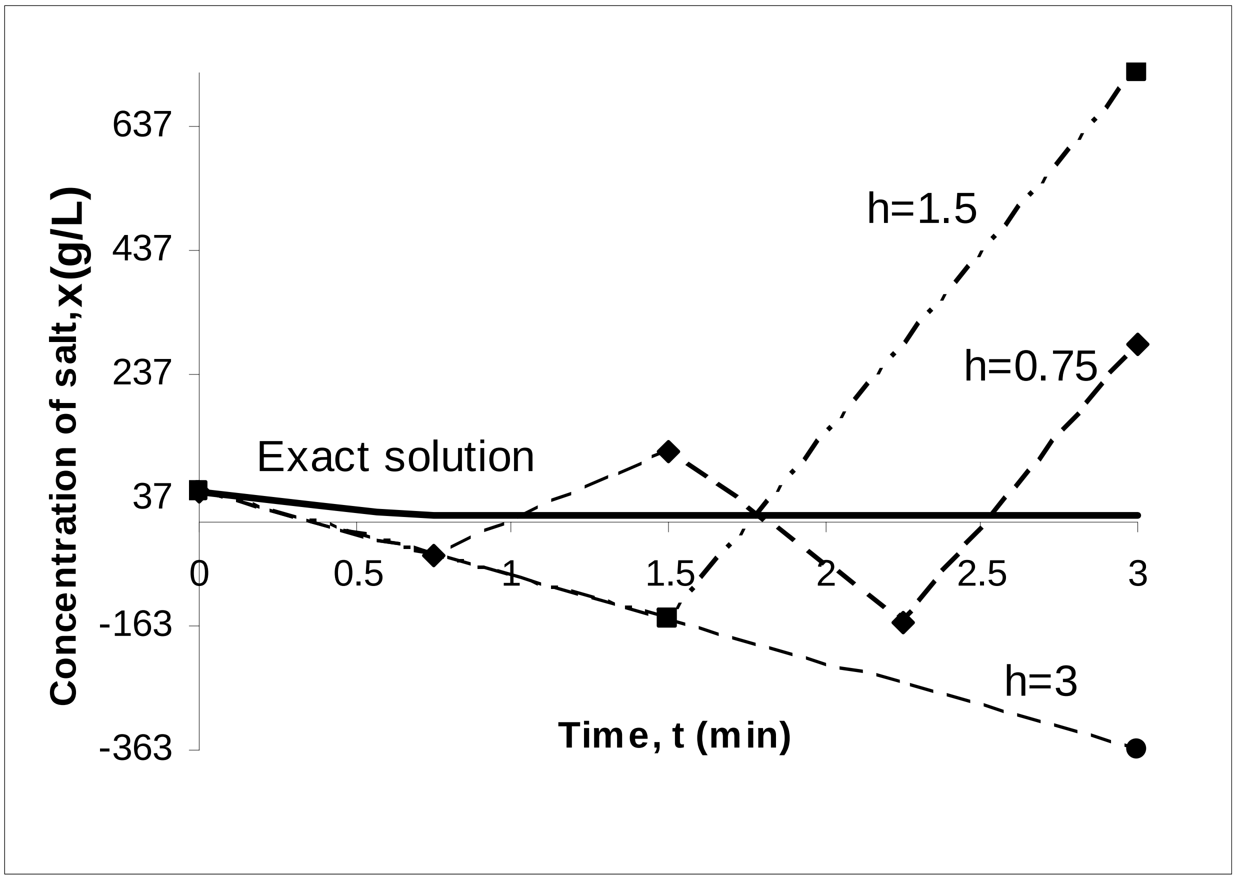

Figure 4. Comparison of Euler's method with exact solution for different step sizes

Effects of step size on Euler's Method

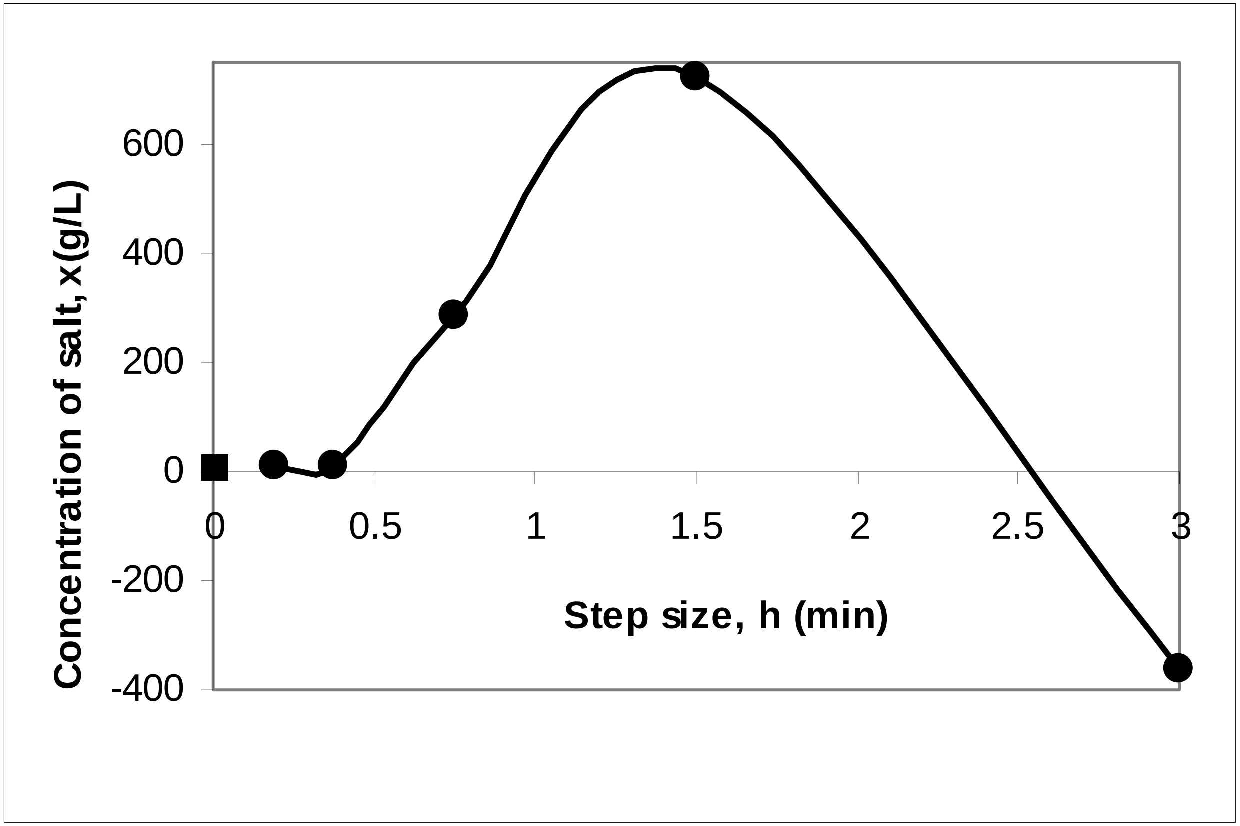

Figure 5. Effect of step size in Euler's method.

Errors in Euler's Method

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i,y_i}\left(x_{i+1}-x_i\right) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i,y_i}\left(x_{i+1}-x_i\right)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i,y_i}\left(x_{i+1}-x_i\right)^3 + \ldots$$

$$y_{i+1} = y_i + f(x_i,y_i) + \frac{1}{2!}f'(x_i,y_i)\left(x_{i+1}-x_i\right)^2 + \frac{1}{3!}f''(x_i,y_i)\left(x_{i+1}-x_i\right)^3 + \ldots$$

As you can see the first two terms of the Taylor series

$y_{i+1} = y_i + f\left(x_i,y_i\right)h$ are the Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'\left(x_i,y_i\right)}{2!}h^2 + \frac{f''\left(x_i,y_i\right)}{3!}h^3 + \ldots \qquad E_t \propto h^2$$