Ordinary Differential Equations

Topic: Runge-Kutta 2nd Order Method

Major: Chemical Engineering

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Runge-Kutta 2nd Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 2nd order method is given by

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

Heun's Method

<u>Heun's method</u>

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + h, y_i + k_1 h\right)$$

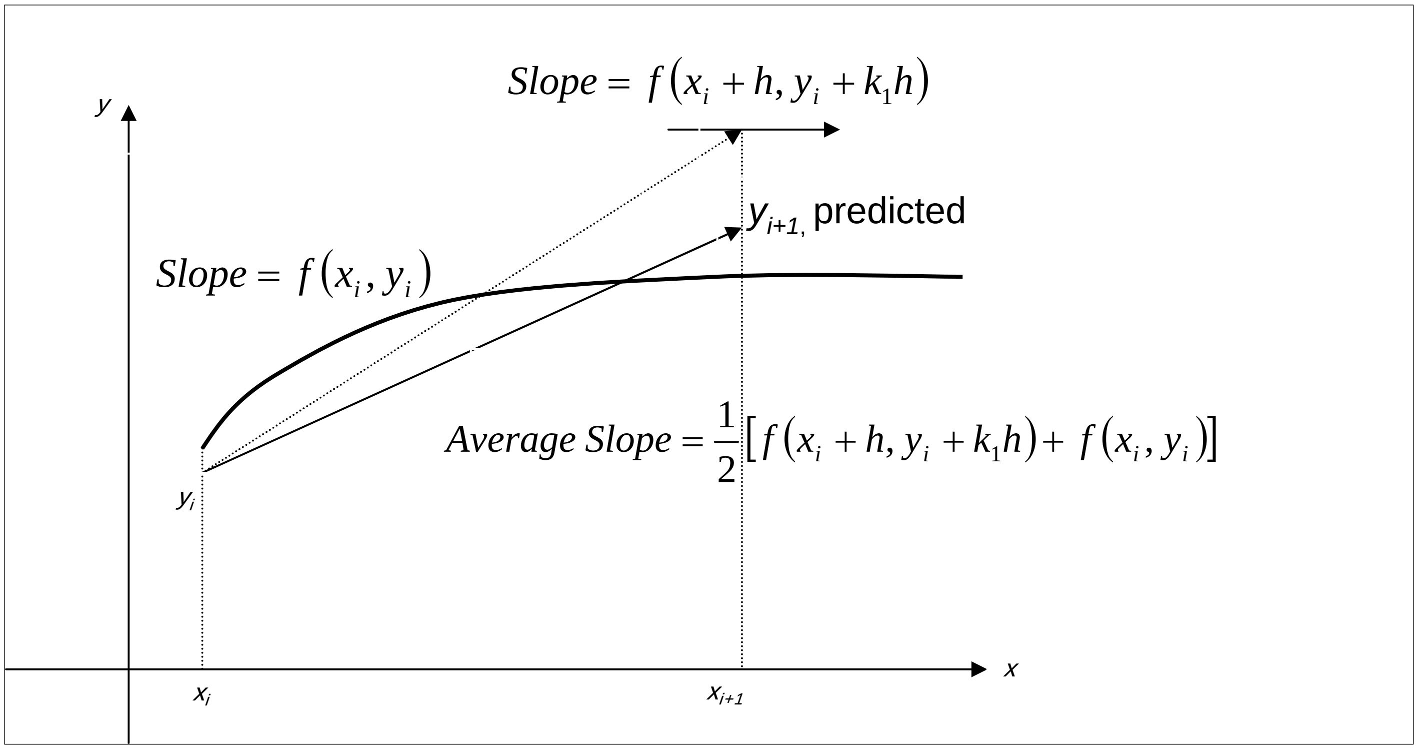

Figure 1. Runge-Kutta 2nd order method (Heun's method)

Midpoint Method

Here $a_2 = 1$ **is chosen, giving**

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

Ralston's Method

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + (\frac{1}{3}k_1 + \frac{2}{3}k_2)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

The concentration of salt, x in a home made soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5x$$

At the initial time, *t = 0*, the salt concentration in the tank is *50g/L.* Using Euler's method and a step size of *h=1.5 min*, what is the salt concentration after *3* minutes.

$$\frac{dx}{dt} = 37.5 - 3.5x$$

$$f(t,x) = 37.5 - 3.5x$$

$$x_{i+1} = x_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

Solution

Step 1: $i = 0, t_0 = 0, x_0 = x(0) = 50$

$$k_1 = f(t_0, x_o) = f(0,50) = 37.5 - 3.5(50) = -137.50$$

$$k_2 = f(t_0 + h, x_0 + k_1 h) = f(0 + 1.5, 50 + (-137.50)1.5) = f(1.5, -156.25)$$

$$= 37.5 - 3.5(-156.25) = 584.38$$

$$x_1 = x_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$= 50 + \left(\frac{1}{2}(-137.50) + \frac{1}{2}(584.38)\right)1.5$$

$$= 50 + (223.44)1.5$$

$$= 385.16 g / L$$

Solution Cont

Step 2: $i=1, t_1 = t_0 + h = 0 + 1.5 = 1.5, x_1 = 385.16 g/L$

$$k_1 = f\left(t_1, x_1\right) = f\left(1.5, 385.16\right) = 37.5 - 3.5(385.16) = -1310.6$$

$$k_2 = f\left(t_1 + h, x_1 + k_1 h\right) = f\left(1.5 + 1.5, 385.16 + \left(-1310.6\right)1.5\right) = f\left(3, -1580.7\right)$$

$$= 37.5 - 3.5(-1580.7) = 5569.9$$

$$x_2 = x_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$= 385.16 + \left(\frac{1}{2}\left(-1310.6\right) + \frac{1}{2}\left(5569.9\right)\right)1.5$$

$$= 385.16 + \left(2129.7\right)1.5$$

$$= 3579.7 g/L$$

Solution Cont

The exact solution of the ordinary differential equation is given by

$$x(t) = 10.714 + 39.286e^{-3.5x}$$

The solution to this nonlinear equation at t=3 minutes is

$$x(3) = 10.715$$

Comparison with exact results

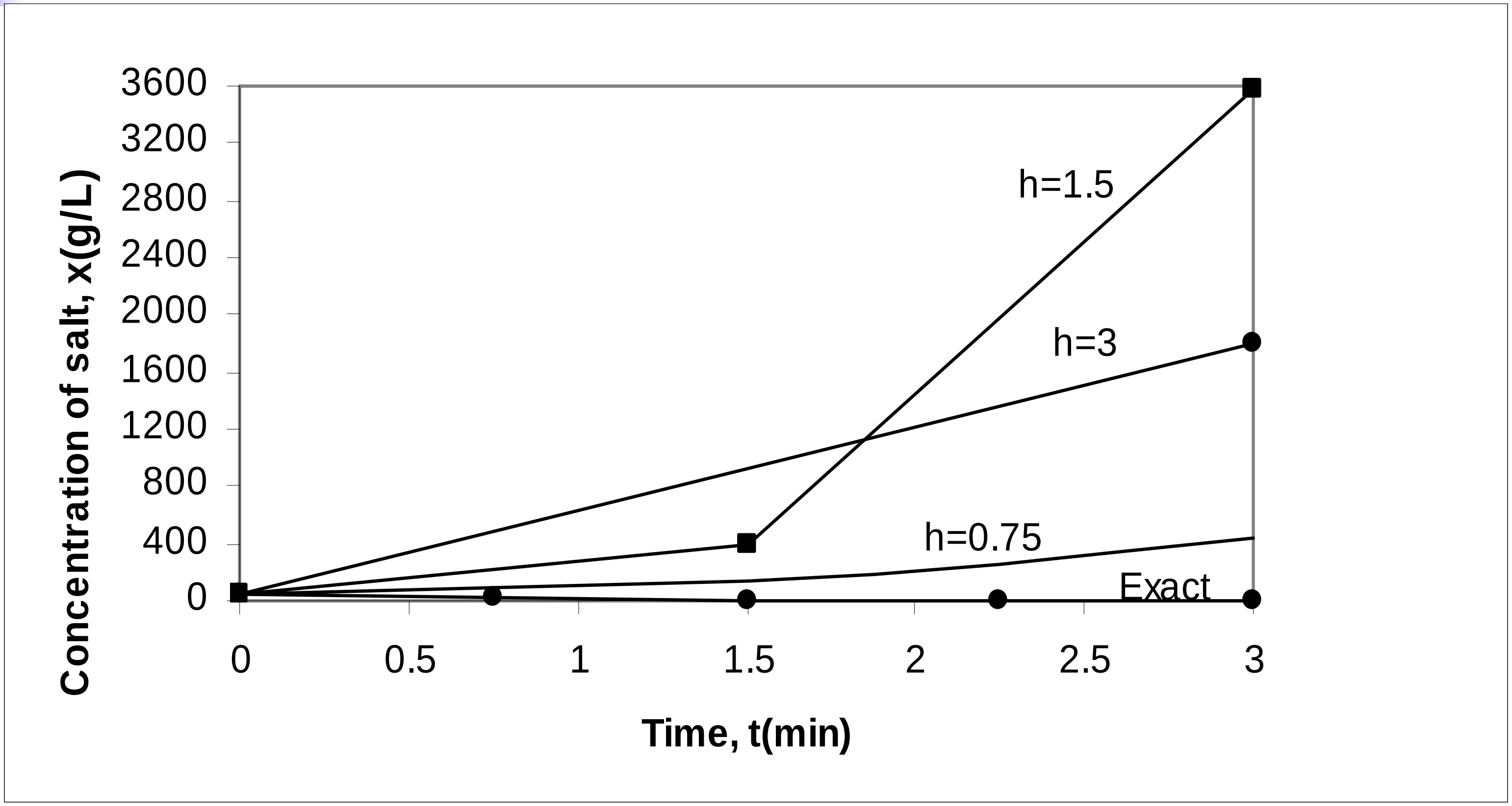

Figure 2. Heun's method results for different step sizes

Effect of step size

Table 1. Effect of step size for Heun's method

Step h Size	$x(3)$	E_t	$\|\in_t\|\%$
3	1803.1	-1792.4	16727
1.5	3579.6	-3568.9	33306
0.75	442.05	-431.34	4025.4
0.375	11.038	-0.32231	3.0079
0.1875	10.718	-.0024979	0.023311

$x(3) = 10.715$ (exact)

Effects of step size on Heun's Method

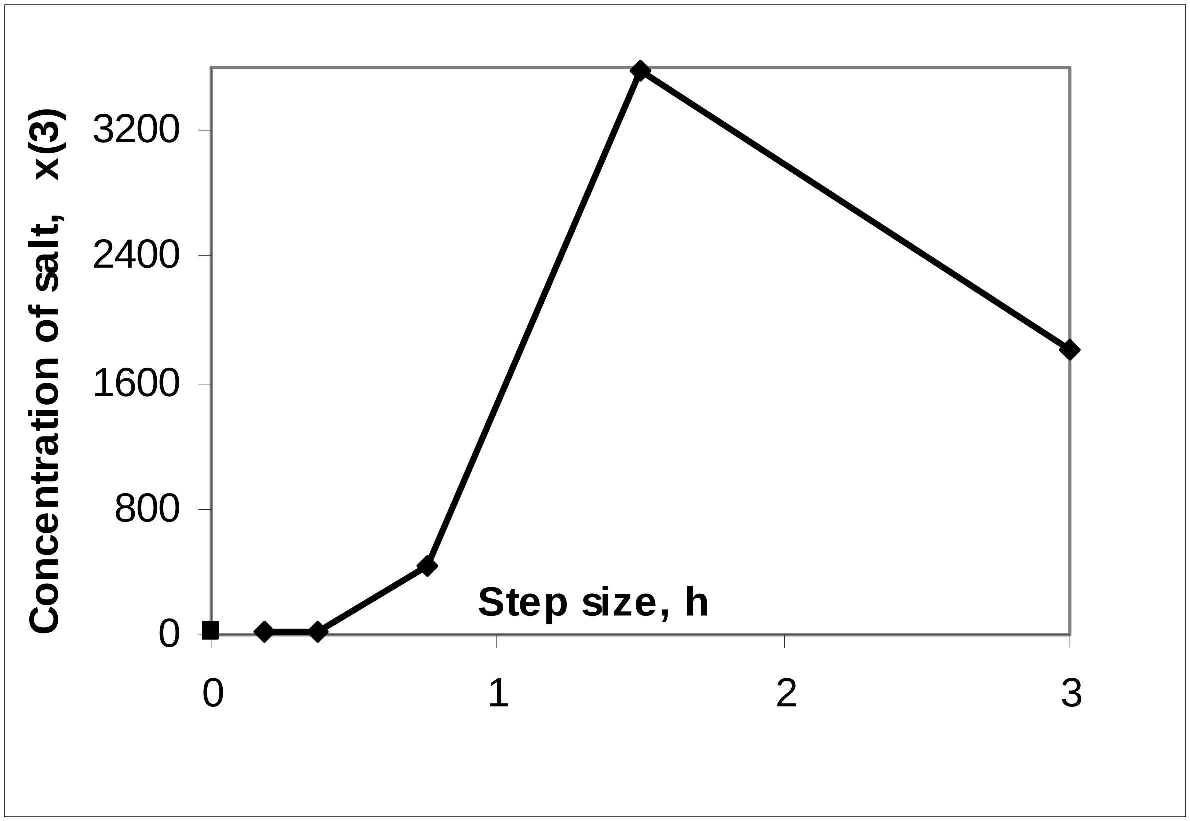

Figure 3. Effect of step size in Heun's method

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$x(3)$			
	Euler	Heun	Midpoint	Ralston
3	-362.50	1803.1	1803.1	1803.1
1.5	720.31	3579.6	3579.6	3579.6
0.75	284.65	442.05	442.05	442.05
0.375	10.718	11.038	11.038	11.038
0.1875	10.714	10.718	10.718	10.718

$x(3) = 10.715$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\left|\in_t\right|\%$			
	Euler	Heun	Midpoint	Ralston
3	3483.0	16727	16727	16727
1.5	6622.2	33306	33306	33306
0.75	2556.5	4025.4	4025.4	4025.4
0.375	0.023249	3.0079	3.0079	3.0079
0.1875	0.010082	0.023311	0.023311	0.023311

$x(3) = 10.715$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

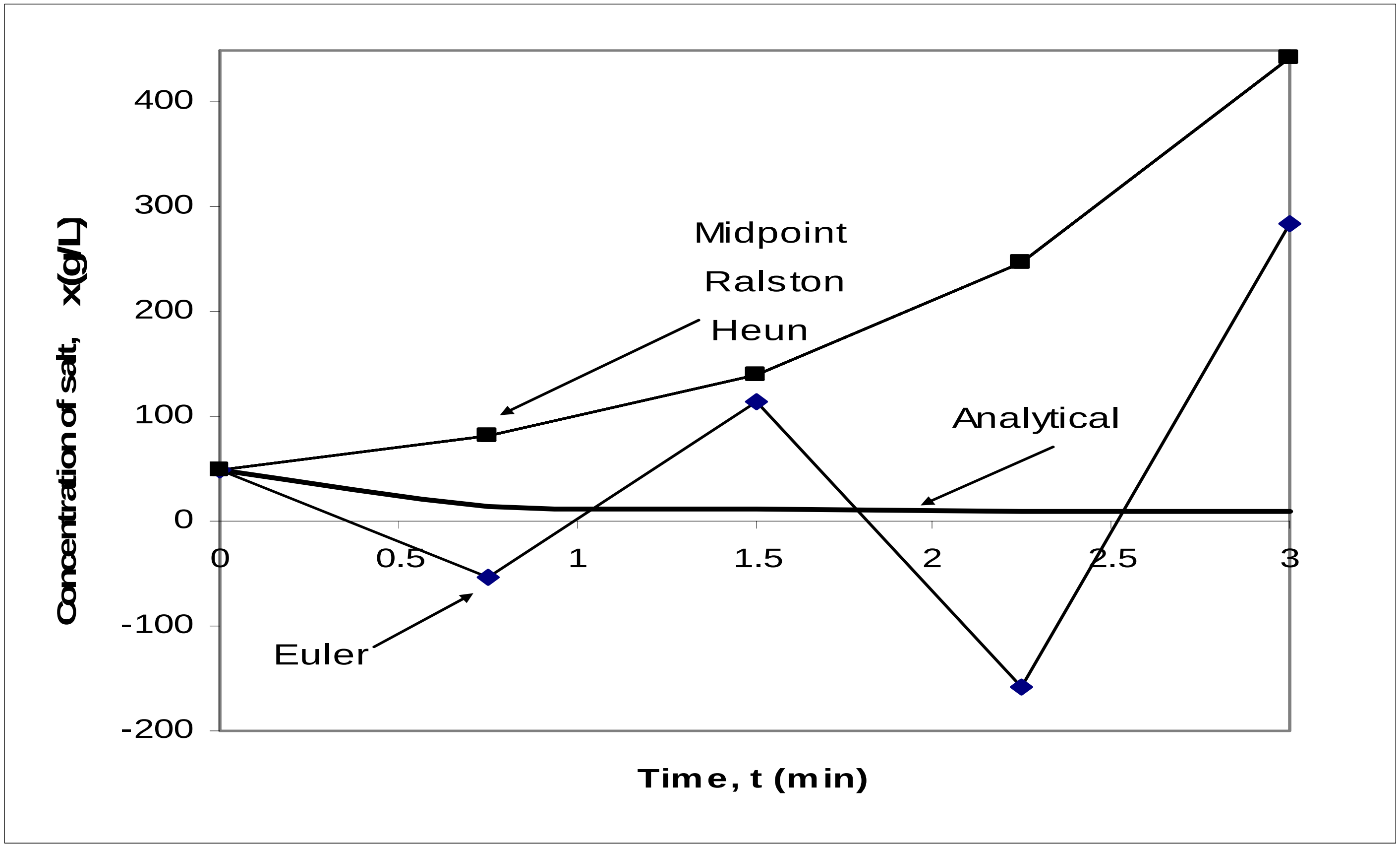

Figure 4. Comparison of Euler and Runge Kutta 2nd order methods with exact results.