Ordinary Differential Equations

Topic: Runge-Kutta 4th Order Method

Major: Chemical Engineering

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Runge-Kutta 4th Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 4th order method is given by

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f\left(x_i + h, y_i + k_3 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

The concentration of salt, x in a home made soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5x$$

At the initial time, *t = 0*, the salt concentration in the tank is *50g/L.* Using Euler's method and a step size of *h=1.5 min*, what is the salt concentration after *3* minutes.

$$\frac{dx}{dt} = 37.5 - 3.5x$$

$$f(t, x) = 37.5 - 3.5x$$

$$x_{i+1} = x_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

Solution

Step 1: $i = 0 \quad t_0 = 0 \quad x_0 = 50 g/L$

$$k_1 = f(t_0, x_0) = f(0,50) = 37.5 - 3.5(50) = -137.50$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, x_0 + \frac{1}{2}k_1 h\right) = f\left(0 + \frac{1}{2}1.5, 50 + \frac{1}{2}(-137.50)1.5\right) = f(0.75, -53.125)$$

$$= 37.5 - 3.5(-53.125) = 223.44$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, x_0 + \frac{1}{2}k_2 h\right) = f\left(0 + \frac{1}{2}1.5, 50 + \frac{1}{2}(223.44)1.5\right) = f(0.75, 217.58)$$

$$= 37.5 - 3.5(217.58) = -724.03$$

$$k_4 = f\left(t_0 + \frac{1}{2}h, x_0 + \frac{1}{2}k_3 h\right) = f\left(0 + \frac{1}{2}1.5, 50 + \frac{1}{2}(-724.03)1.5\right) = f(0.75, -493.02)$$

$$= 37.5 - 3.5(-493.02) = 1763.1$$

Solution Cont

$$x_1 = x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 50 + \frac{1}{6}\left(-137.50 + 2(223.44) + 2(-724.03) + (1763.1)\right)1.5$$

$$= 50 + \frac{1}{6}(624.42)1.5$$

$$= 206.11 g/L$$

x_1 is the approximate concentration of salt at

$$t = t_1 = t_0 + h = 0 + 1.5 = 1.5$$

$$x_1 = x(1.5) \approx 206.11 g/L$$

Solution Cont

Step 2: $i=1, t_1=1.5, x_1=206.11 g/L$

$$k_1 = f(t_1, x_1) = f(1.5, 206.11) = 37.5 - 3.5(206.11) = -683.89$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, x_1 + \frac{1}{2}k_1 h\right) = f\left(1.5 + \frac{1}{2}1.5, 206.11 + \frac{1}{2}(-683.89)1.5\right) = f(2.25, -306.80)$$

$$= 37.5 - 3.5(-306.80) = 1111.3$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, x_1 + \frac{1}{2}k_2 h\right) = f\left(1.5 + \frac{1}{2}1.5, 206.11 + \frac{1}{2}(1111.3)1.5\right) = f(2.25, 1039.6)$$

$$= 37.5 - 3.5(1039.6) = -3601.1$$

$$k_4 = f\left(t_1 + \frac{1}{2}h, x_1 + k_3 h\right) = f\left(1.5 + \frac{1}{2}1.5, 206.11 + \frac{1}{2}(-3601.1)1.5\right) = f(2.25, -2494.7)$$

$$= 37.5 - 3.5(-2494.7) = 8769.0$$

Solution Cont

$$x_2 = x_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 206.11 + \frac{1}{6}\left(-683.89 + 2\left(1111.3\right) + 2\left(-3601.1\right) + 8769.0\right)1.5$$

$$= 206.11 + \frac{1}{6}\left(3105.5\right)1.5$$

$$= 982.49 g / L$$

x_2 is the approximate concentration of salt at

$t_2 = t_1 + h$ = $1.5 + 1.5 = 3\min$

$$x_2 = x\left(3\right) \approx 982.49 g / L$$

Solution Cont

The exact solution of the ordinary differential equation is given by

$$x(t) = 10.714 + 39.286e^{-3.5x}$$

The solution to this nonlinear equation at t=3 minutes is

$$x(3) = 10.715$$

Comparison with exact results

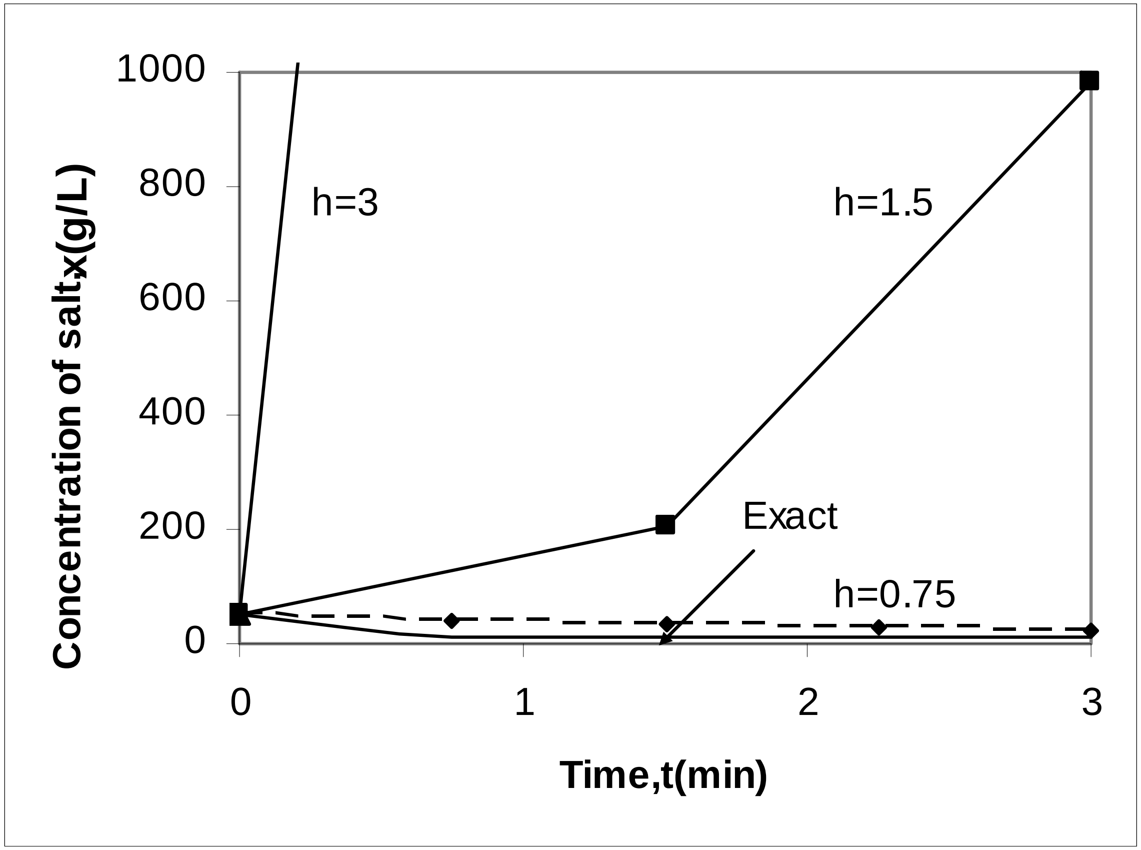

Figure 1. Comparison of Runge-Kutta 4th order method with exact solution

Effect of step size

Table 1. Value of concentration of salt at 3 minutes for different step sizes

Step h	$x(3)$	E_t	$\lvert\epsilon_t\rvert\,\%$
3	14120	-14109	131670
1.5	11455	-11444	106800
0.75	25.559	-14.843	138.52
0.375	10.717	-0.0012112	0.011303
0.1875	10.715	0	0

$$x(3) = 10.715 \qquad \text{(exact)}$$

Effects of step size on Runge-Kutta 4th Order Method

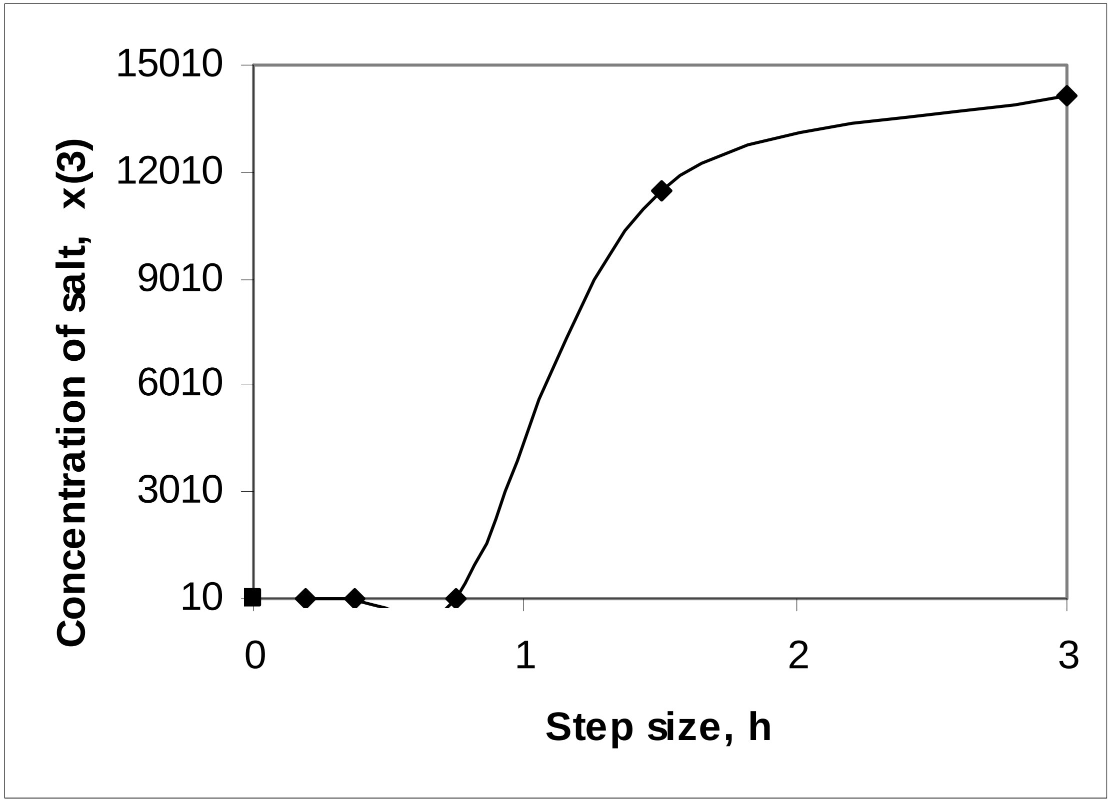

Figure 2. Effect of step size in Runge-Kutta 4th order method

Comparison of Euler and Runge-Kutta Methods

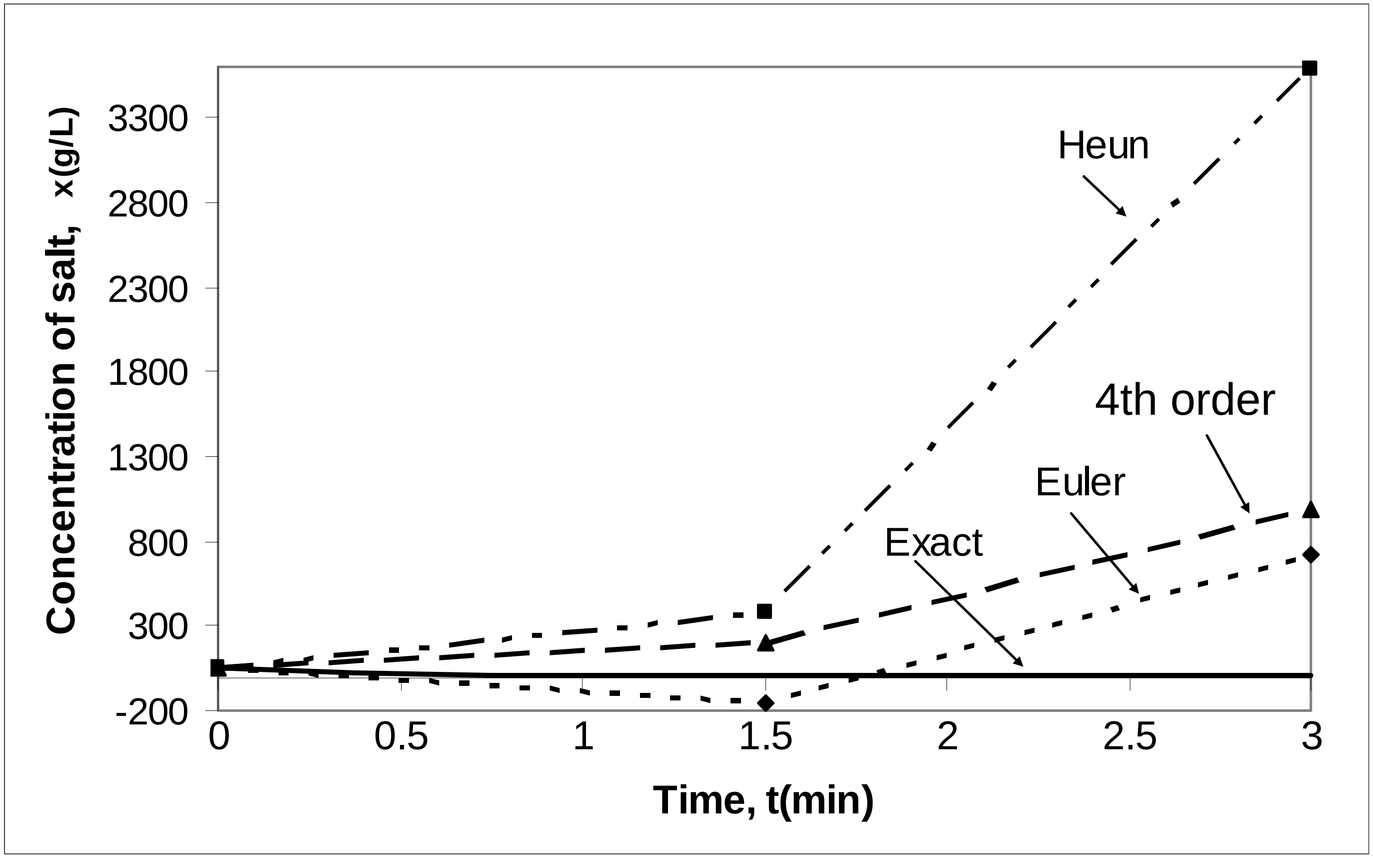

Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.