Chapter 08.02
Euler's Method for Ordinary Differential Equations

After reading this chapter, you should be able to:

1. *develop Euler's Method for solving ordinary differential equations,*
2. *determine how the step size affects the accuracy of a solution,*
3. *derive Euler's formula from Taylor series, and*
4. *use Euler's method to find approximate values of integrals.*

What is Euler's method?

Euler's method is a numerical technique to solve ordinary differential equations of the form

$$\frac{dy}{dx} = f(x,y), y(0) = y_0 \quad (1)$$

So only first order ordinary differential equations can be solved by using Euler's method. In another chapter we will discuss how Euler's method is used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations. How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x,y), \ y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x,y) = 1.3e^{-x} - 2y$$

Example 2

Rewrite

$$e^y \frac{dy}{dx} + x^2 y^2 = 2\sin(3x),\ \ y(0) = 5$$

in

$$\frac{dy}{dx} = f(x,y),\ \ y(0) = y_0 \text{ form.}$$

Solution

$$e^y \frac{dy}{dx} + x^2 y^2 = 2\sin(3x),\ \ y(0) = 5$$

$$\frac{dy}{dx} = \frac{2\sin(3x) - x^2 y^2}{e^y},\ \ y(0) = 5$$

In this case

$$f(x,y) = \frac{2\sin(3x) - x^2 y^2}{e^y}$$

Derivation of Euler's method

At $x = 0$, we are given the value of $y = y_0$. Let us call $x = 0$ as x_0. Now since we know the slope of y with respect to x, that is, $f(x,y)$, then at $x = x_0$, the slope is $f(x_0, y_0)$. Both x_0 and y_0 are known from the initial condition $y(x_0) = y_0$.

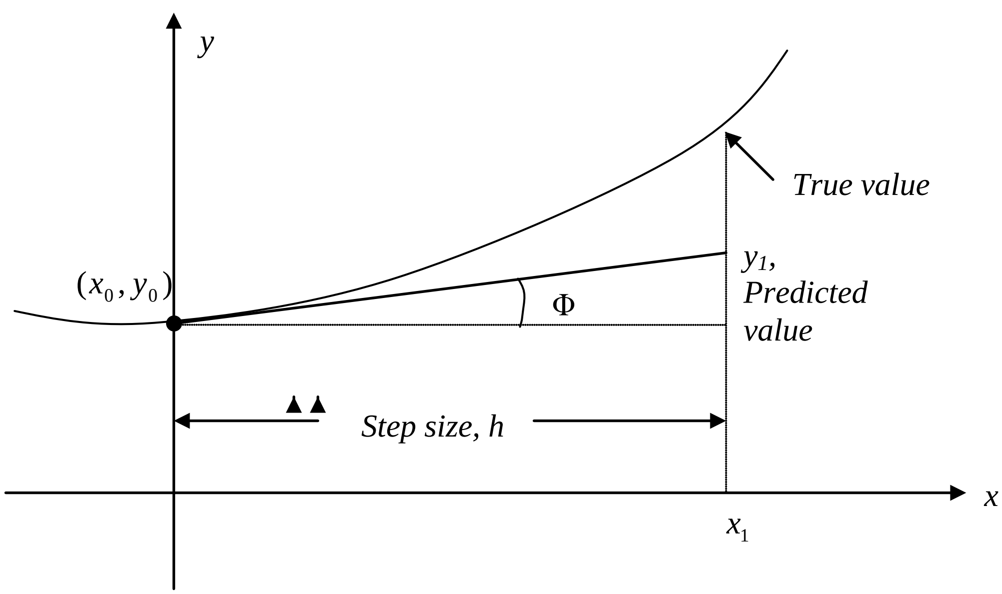

Figure 1 Graphical interpretation of the first step of Euler's method.

So the slope at $x = x_0$ as shown in Figure 1 is

$$\text{Slope} = \frac{Rise}{Run}$$
$$= \frac{y_1 - y_0}{x_1 - x_0}$$
$$= f(x_0, y_0)$$

From here

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

Calling $x_1 - x_0$ the step size h, we get

$$y_1 = y_0 + f(x_0, y_0)h \qquad (2)$$

One can now use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2, and that would be the predicted value at x_2, given by

$$y_2 = y_1 + f(x_1, y_1)h$$
$$x_2 = x_1 + h$$

Based on the above equations, if we now know the value of $y = y_i$ at x_i, then

$$y_{i+1} = y_i + f(x_i, y_i)h \qquad (3)$$

This formula is known as Euler's method and is illustrated graphically in Figure 2. In some books, it is also called the Euler-Cauchy method.

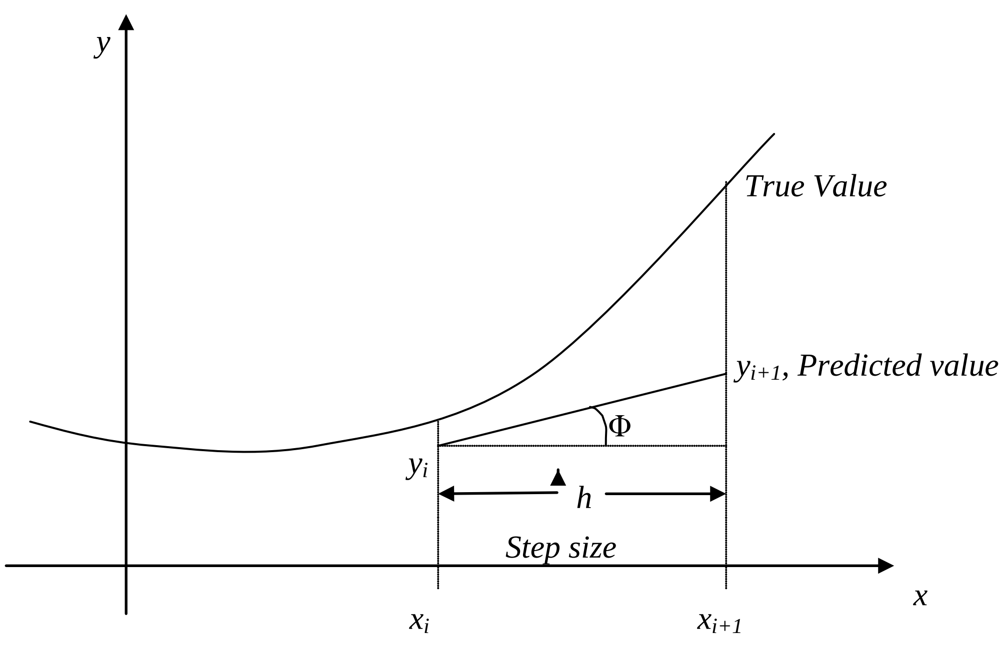

Figure 2 General graphical interpretation of Euler's method.

Example 3

The concentration of salt x in a home made soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5x$$

At the initial time, $t = 0$, the salt concentration in the tank is 50 g/L. Using Euler's method and a step size of $h = 1.5\ \text{min}$, what is the salt concentration after 3 minutes?

Solution

$$\frac{dx}{dt} = 37.5 - 3.5x$$

$$f(t, x) = 37.5 - 3.5x$$

The Euler's method reduces to

$$x_{i+1} = x_i + f(t_i, x_i)h$$

For $i = 0$, $t_0 = 0$, $x_0 = 50$

$$\begin{aligned} x_1 &= x_0 + f(t_0, x_0)h \\ &= 50 + f(0,50)1.5 \\ &= 50 + (37.5 - 3.5(50))1.5 \\ &= 50 + (-137.5)1.5 \\ &= -156.25\ \text{g/L} \end{aligned}$$

x_1 is the approximate concentration of salt at

$$t = t_1 = t_0 + h = 0 + 1.5 = 1.5\ \text{min}$$

$$x(1.5) \approx x_1 = -156.25\ \text{g/L}$$

For $i = 1$, $t_1 = 1.5$, $x_1 = -156.25$

$$\begin{aligned} x_2 &= x_1 + f(t_1, x_1)h \\ &= -156.25 + f(1.5, -156.25)1.5 \\ &= -156.25 + (37.5 - 3.5(-156.25))1.5 \\ &= -156.25 + (584.38)1.5 \\ &= 720.31\ \text{g/L} \end{aligned}$$

x_2 is the approximate concentration of salt at

$$t = t_2 = t_1 + h = 1.5 + 1.5 = 3\ \text{min}$$

$$x(3) \approx x_2 = 720.31\ \text{g/L}$$

Figure 3 compares the exact solution with the numerical solution from Euler's method for the step size of $h = 1.5$.

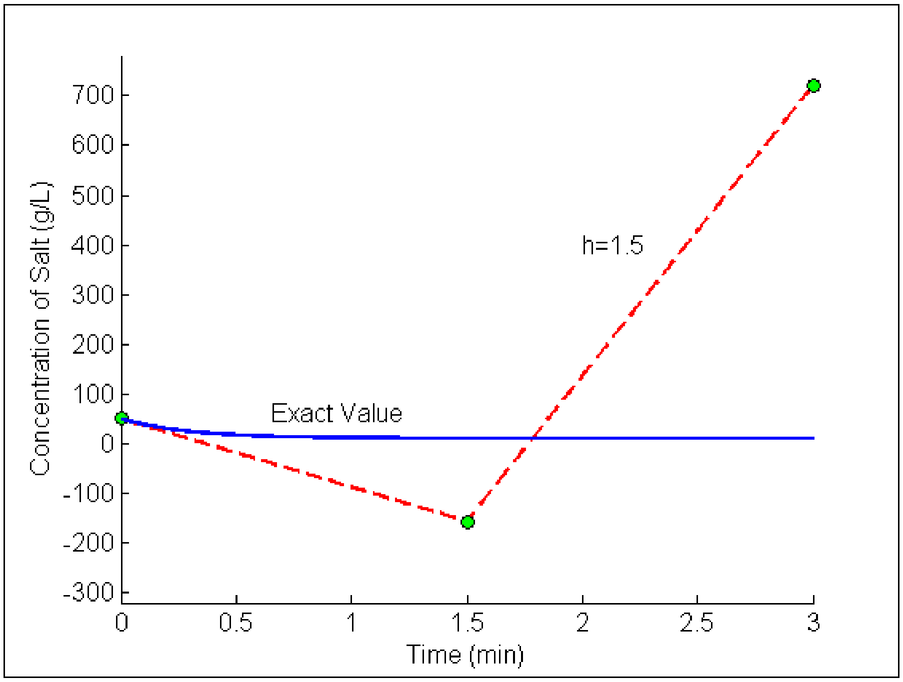

Figure 3 Comparing exact and Euler's method.

The problem was solved again using smaller step sizes. The results are given below in Table 1.

Table 1 Concentration of salt at 3 minutes as a function of step size, h.

step size, h	$x(3)$	E_t	$\|\in_t\|\%$
3	–362.5	373.22	3483.0
1.5	720.31	–709.60	6622.2
0.75	284.65	–273.93	2556.5
0.375	10.718	–0.0024912	0.023249
0.1875	10.714	0.0010803	0.010082

Figure 4 shows how the concentration of salt varies as a function of time for different step sizes.

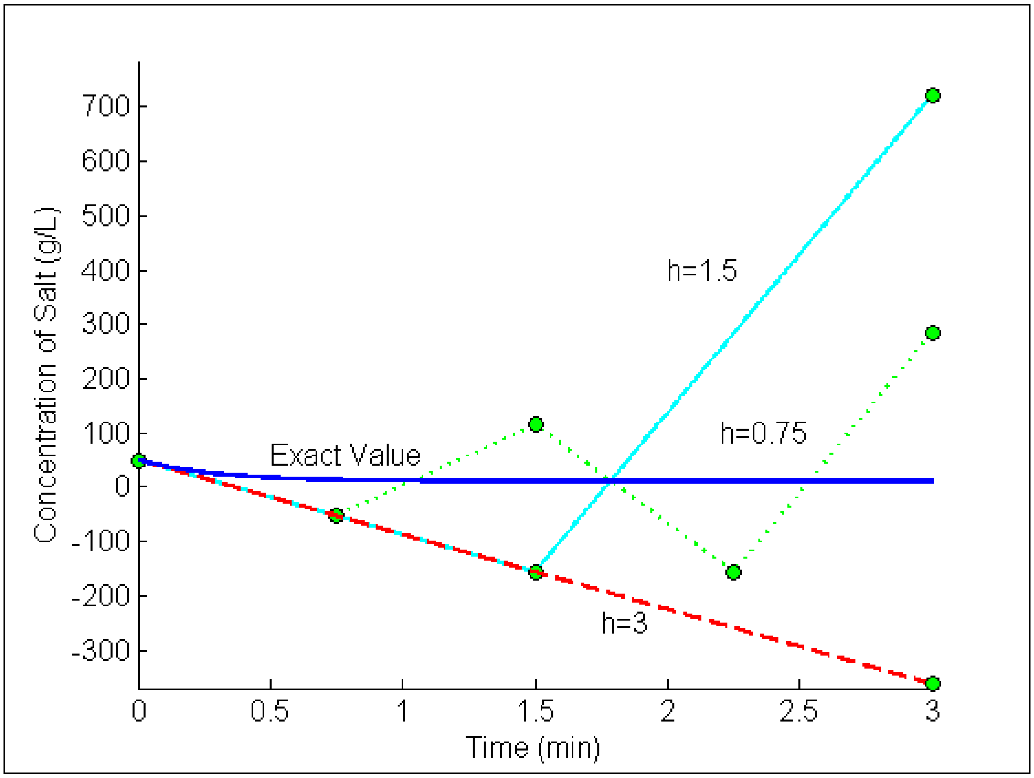

Figure 4 Comparison of Euler's method with exact solution for different step sizes.

While the values of the calculated concentration of salt at $t = 3$ min as a function of step size are plotted in Figure 5.

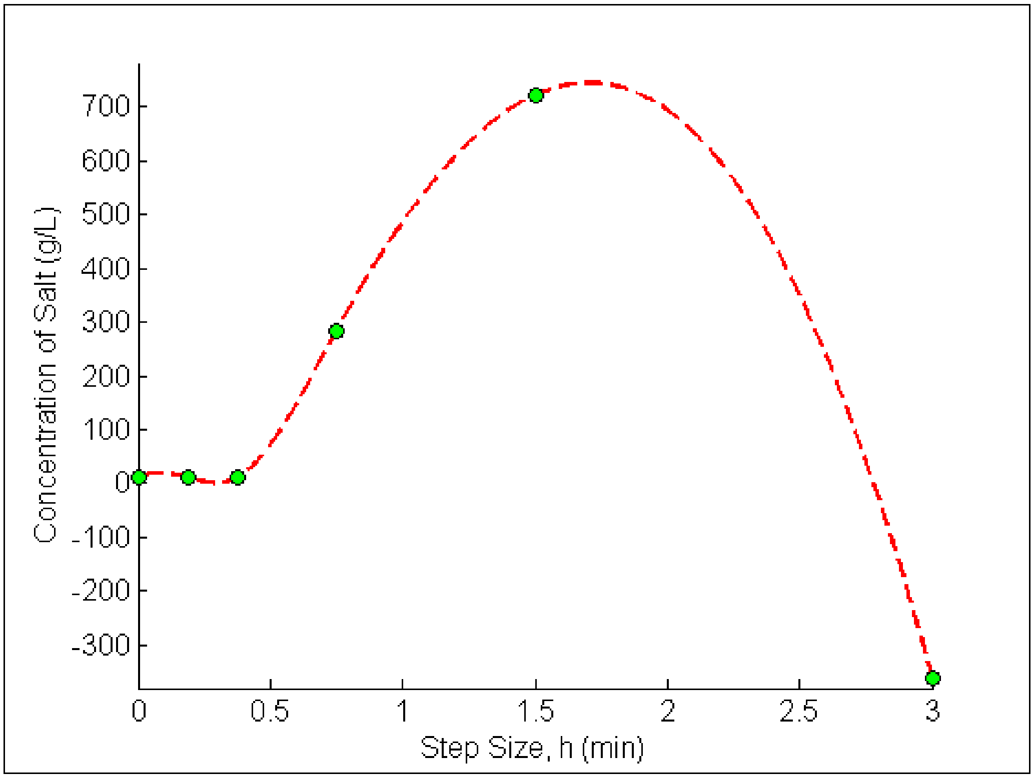

Figure 5 Effect of step size in Euler's method.

The exact solution of the ordinary differential equation is given by

$$x(t) = 10.714 + 39.286e^{-3.5t}$$

The solution to this nonlinear equation at $t = 3$ min is

$$x(3) = 10.715 \text{ g/L}$$

It can be seen that Euler's method has large errors. This can be illustrated using the Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i, y_i}(x_{i+1} - x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i, y_i}(x_{i+1} - x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i, y_i}(x_{i+1} - x_i)^3 + \ldots \quad (5)$$

$$= y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!}f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!}f''(x_i, y_i)(x_{i+1} - x_i)^3 + \ldots \quad (6)$$

As you can see the first two terms of the Taylor series

$$y_{i+1} = y_i + f(x_i, y_i)h$$

are Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!}h^2 + \frac{f''(x_i, y_i)}{3!}h^3 + \ldots \quad (7)$$

The true error hence is approximately proportional to the square of the step size, that is, as the step size is halved, the true error gets approximately quartered. However from Table 1, we see that as the step size gets halved, the true error only gets approximately halved. This is because the true error, being proportioned to the square of the step size, is the local truncation error, that is, error from one point to the next. The global truncation error is however proportional only to the step size as the error keeps propagating from one point to another.

Can one solve a definite integral using numerical methods such as Euler's method of solving ordinary differential equations?

Let us suppose you want to find the integral of a function $f(x)$

$$I = \int_a^b f(x)dx .$$

Both fundamental theorems of calculus would be used to set up the problem so as to solve it as an ordinary differential equation.

The first fundamental theorem of calculus states that if f is a continuous function in the interval [a,b], and F is the antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

The second fundamental theorem of calculus states that if f is a continuous function in the open interval D, and a is a point in the interval D, and if

$$F(x) = \int_a^x f(t)dt$$

then

$$F'(x) = f(x)$$

at each point in D.

Asked to find $\int_a^b f(x)dx$, we can rewrite the integral as the solution of an ordinary differential equation (here is where we are using the second fundamental theorem of calculus)

$$\frac{dy}{dx} = f(x),\ y(a) = 0,$$

where then $y(b)$ (here is where we are using the first fundamental theorem of calculus) will give the value of the integral $\int_a^b f(x)dx$.

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Euler's Method for ordinary differential equations
Summary	Textbook notes on Euler's method for solving ordinary differential equations
Major	Chemical Engineering
Authors	Autar Kaw
Last Revised	September 1, 2009
Web Site	http://numericalmethods.eng.usf.edu