Chapter 08.04
Runge-Kutta 4th Order Method for Ordinary Differential Equations

After reading this chapter, you should be able to

1. *develop Runge-Kutta 4th order method for solving ordinary differential equations,*
2. *find the effect size of step size has on the solution,*
3. *know the formulas for other versions of the Runge-Kutta 4th order method*

What is the Runge-Kutta 4th order method?

Runge-Kutta 4$^{\text{th}}$ order method is a numerical technique used to solve ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

So only first order ordinary differential equations can be solved by using the Runge-Kutta 4$^{\text{th}}$ order method. In other sections, we have discussed how Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \ y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx}+2y=1.3e^{-x},\ y(0)=5$$

$$\frac{dy}{dx}=1.3e^{-x}-2y,\ y(0)=5$$

In this case

$$f(x,y)=1.3e^{-x}-2y$$

Example 2

Rewrite

$$e^{y}\frac{dy}{dx}+x^2y^2=2\sin(3x),\ \ y(0)=5$$

in

$$\frac{dy}{dx}=f(x,y),\ \ y(0)=y_0 \text{ form.}$$

Solution

$$e^{y}\frac{dy}{dx}+x^2y^2=2\sin(3x),\ \ y(0)=5$$

$$\frac{dy}{dx}=\frac{2\sin(3x)-x^2y^2}{e^{y}},\ \ y(0)=5$$

In this case

$$f(x,y)=\frac{2\sin(3x)-x^2y^2}{e^{y}}$$

The Runge-Kutta 4th order method is based on the following

$$y_{i+1}=y_i+(a_1k_1+a_2k_2+a_3k_3+a_4k_4)h \tag{1}$$

where knowing the value of $y=y_i$ at x_i, we can find the value of $y=y_{i+1}$ at x_{i+1}, and

$$h=x_{i+1}-x_i$$

Equation (1) is equated to the first five terms of Taylor series

$$y_{i+1}=y_i+\frac{dy}{dx}\bigg|_{x_i,y_i}(x_{i+1}-x_i)+\frac{1}{2!}\frac{d^2y}{dx^2}\bigg|_{x_i,y_i}(x_{i+1}-x_i)^2+\frac{1}{3!}\frac{d^3y}{dx^3}\bigg|_{x_i,y_i}(x_{i+1}-x_i)^3 +\frac{1}{4!}\frac{d^4y}{dx^4}\bigg|_{x_i,y_i}(x_{i+1}-x_i)^4 \tag{2}$$

Knowing that $\frac{dy}{dx}=f(x,y)$ and $x_{i+1}-x_i=h$

$$y_{i+1}=y_i+f(x_i,y_i)h+\frac{1}{2!}f'(x_i,y_i)h^2+\frac{1}{3!}f''(x_i,y_i)h^3+\frac{1}{4!}f'''(x_i,y_i)h^4 \tag{3}$$

Based on equating Equation (2) and Equation (3), one of the popular solutions used is

$$y_{i+1}=y_i+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h \tag{4}$$

$$k_1 = f\left(x_i, y_i\right) \quad (5a)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) \quad (5b)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right) \quad (5c)$$

$$k_4 = f\left(x_i + h, y_i + k_3 h\right) \quad (5d)$$

Example 3

The concentration of salt x in a home made soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5x$$

At the initial time, $t = 0$, the salt concentration in the tank is 50 g/L Using Runge-Kutta 4th order method and a step size of, $h = 1.5 \text{ min}$, what is the salt concentration after 3 minutes?

Solution

$$\frac{dx}{dt} = 37.5 - 3.5x$$

$$f\left(t, x\right) = 37.5 - 3.5x$$

$$x_{i+1} = x_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

For $i = 0$, $t_0 = 0$, $x_0 = 50$

$$\begin{aligned} k_1 &= f\left(t_0, x_0\right) \\ &= f\left(0,50\right) \\ &= 37.5 - 3.5(50) \\ &= -137.5 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(t_0 + \frac{1}{2}h, x_0 + \frac{1}{2}k_1 h\right) \\ &= f\left(0 + \frac{1}{2}1.5, 50 + \frac{1}{2}\left(-137.5\right)1.5\right) \\ &= f\left(0.75, -53.125\right) \\ &= 37.5 - 3.5(-53.125) \\ &= 223.44 \end{aligned}$$

$$\begin{aligned} k_3 &= f\left(t_0 + \frac{1}{2}h, x_0 + \frac{1}{2}k_2 h\right) \\ &= f\left(0 + \frac{1}{2}1.5, 50 + \frac{1}{2}\left(223.44\right)1.5\right) \\ &= f\left(0.75, 217.58\right) \\ &= 37.5 - 3.5(217.58) \\ &= -724.02 \end{aligned}$$

$$\begin{aligned} k_4 &= f\left(t_0 + h, x_0 + k_3 h\right) \\ &= f\left(0 + 1.5, 50 + \left(-724.03\right)1.5\right) \\ &= f\left(1.5, -1036.0\right) \\ &= 37.5 - 3.5(-1036.0) \\ &= 3663.6 \end{aligned}$$

$$\begin{aligned} x_1 &= x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\ &= 50 + \frac{1}{6}\left(-137.5 + 2\left(223.44\right) + 2\left(-724.02\right) + \left(3663.6\right)\right)1.5 \\ &= 50 + \frac{1}{6}\left(2525.0\right)1.5 \\ &= 681.24 \text{ g/L} \end{aligned}$$

x_1 is the approximate concentration of salt at

$$t = t_1 = t_0 + h = 0 + 1.5 = 1.5$$

$$x\left(1.5\right) \approx x_1 = 681.24 \text{ g/L}$$

For $i = 1$, $t_1 = 1.5$, $x_1 = 681.24$

$$\begin{aligned} k_1 &= f\left(t_1, x_1\right) \\ &= f\left(1.5, 681.24\right) \\ &= 37.5 - 3.5(681.24) \\ &= -2346.8 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(t_1 + \frac{1}{2}h, x_1 + \frac{1}{2}k_1 h\right) \\ &= f\left(1.5 + \frac{1}{2}1.5, 681.24 + \frac{1}{2}\left(-2346.8\right)1.5\right) \\ &= f\left(2.25, -1078.9\right) \\ &= 37.5 - 3.5(-1078.9) \\ &= 3813.6 \end{aligned}$$

$$\begin{aligned} k_3 &= f\left(t_1 + \frac{1}{2}h, x_1 + \frac{1}{2}k_2 h\right) \\ &= f\left(1.5 + \frac{1}{2}1.5, 681.24 + \frac{1}{2}\left(3813.6\right)1.5\right) \\ &= f\left(2.25, 3541.4\right) \\ &= 37.5 - 3.5(3541.4) \\ &= -12358 \end{aligned}$$

$$\begin{aligned} k_4 &= f\left(t_1 + h, x_1 + k_3 h\right) \\ &= f\left(1.5 + 1.5, 681.24 + \left(-12358\right)1.5\right) \\ &= f\left(3, -17855\right) \\ &= 37.5 - 3.5(-17855) \end{aligned}$$

$$= 62530$$

$$x_2 = x_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 681.24 + \frac{1}{6}(-2346.8 + 2(3813.6) + 2(-12358) + 62530)1.5$$

$$= 681.24 + \frac{1}{6}(43096)1.5$$

$$= 11455 \text{ g/L}$$

x_2 is the approximate concentration of salt at

$$t_2 = t_1 + h = 1.5 + 1.5 = 3\,\text{min}$$

$$x(3) \approx x_2 = 11455 \text{ g/L}$$

Figure 1 compares the exact solution with the numerical solution using Runge-Kutta 4th order method using different step sizes.

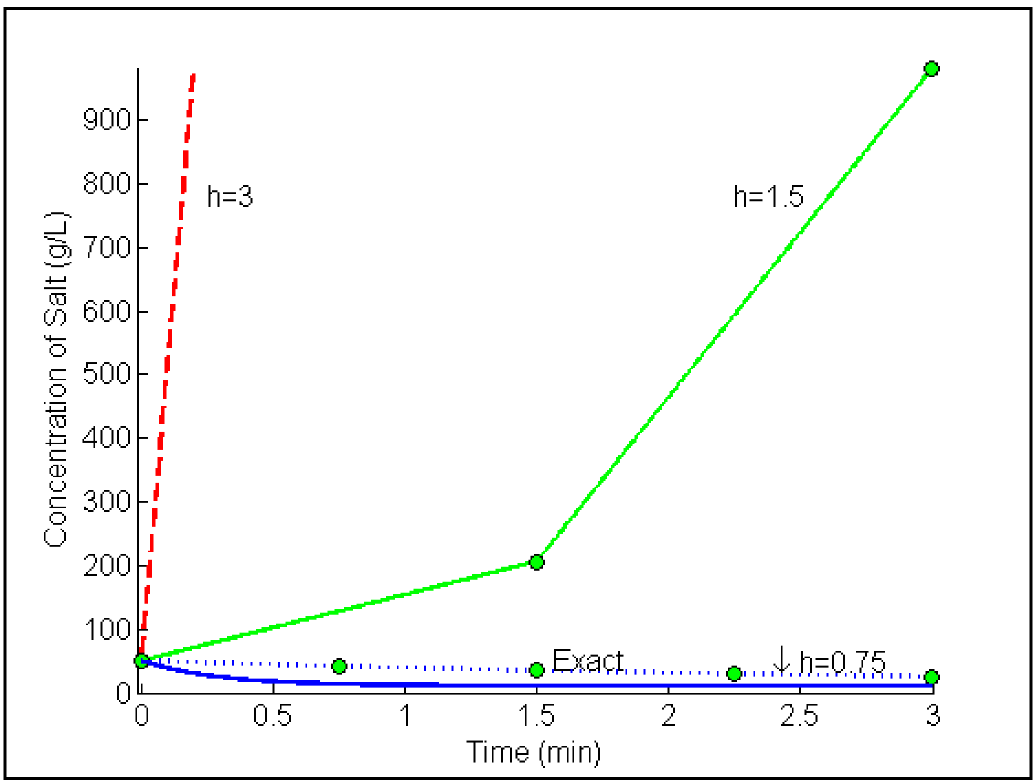

Figure 1 Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.

Table 1 and Figure 2 show the effect of step size on the value of the calculated temperature at $t = 3$ min.

Table 1 Value of concentration of salt at 3 minutes for different step sizes.

Step size, h	$x(3)$	E_t	$\lvert\in_t\rvert$ %
3	14120	−14109	131680
1.5	11455	−11444	106800
0.75	25.559	−14.843	138.53
0.375	10.717	−0.0014969	0.013969
0.1875	10.715	−0.00031657	0.0029544

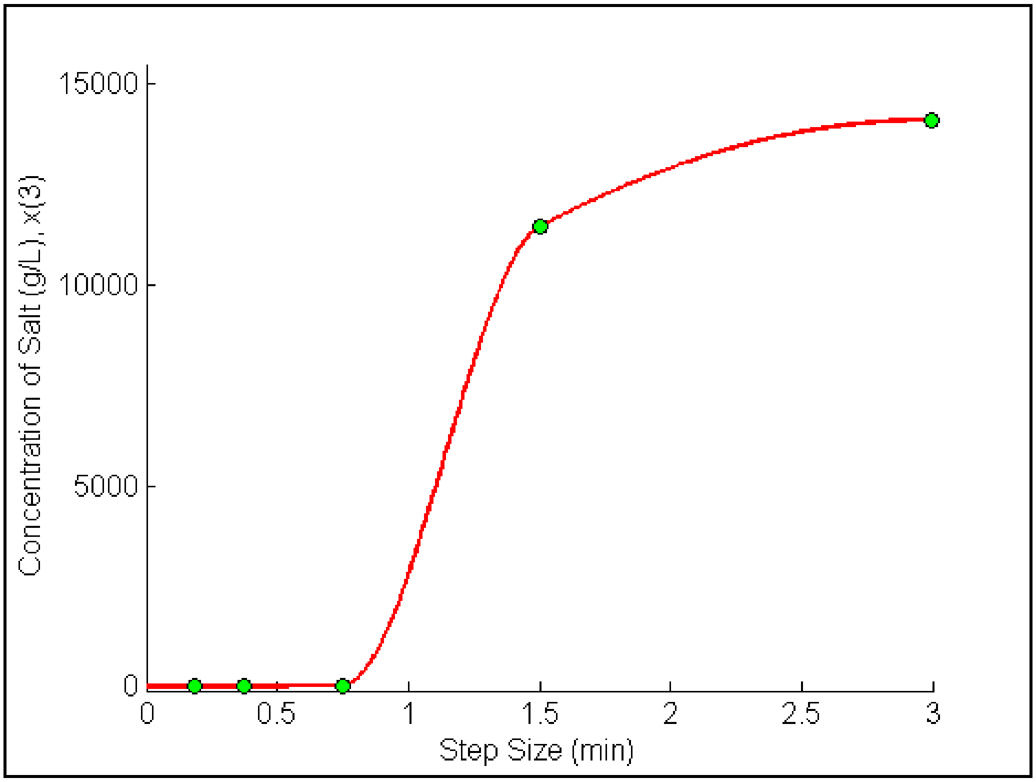

Figure 2 Effect of step size in Runge-Kutta 4^{th} order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1^{st} order method), Heun's method (Runge-Kutta 2^{nd} order method) and Runge-Kutta 4^{th} order method.

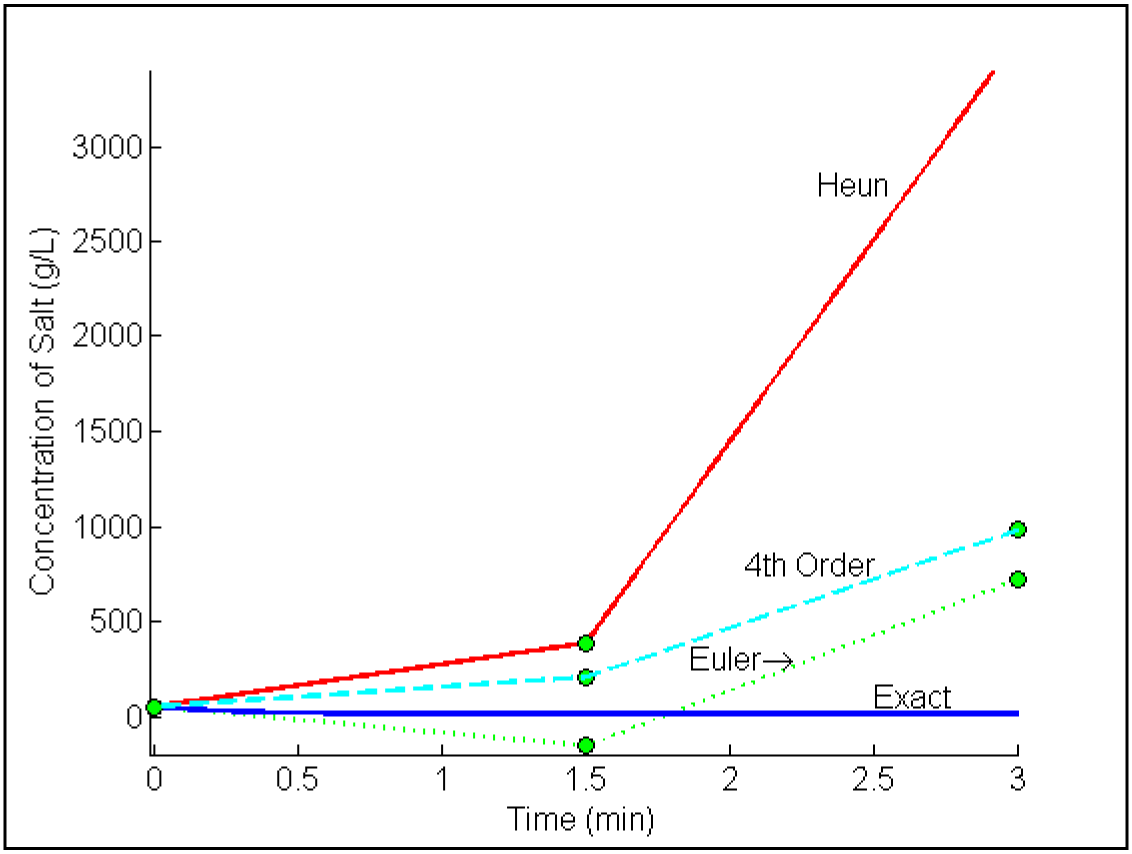

Figure 3 Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

The exact solution of the ordinary differential equation is given by

$$x(t) = 10.714 + 39.286e^{-3.5t}$$

The solution to this nonlinear equation at $t = 3\text{min}$ is

$$x(3) = 10.715 \text{ g/L}$$

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Runge-Kutta 4th order method
Summary	Textbook notes on the Runge-Kutta 4th order method for solving ordinary differential equations.
Major	Chemcial Engineering
Authors	Autar Kaw
Last Revised	September 2, 2009
Web Site	http://numericalmethods.eng.usf.edu