

Chapter 02.03

Differentiation of Discrete Functions

After reading this chapter, you should be able to:

- 1. find approximate values of the first derivative of functions that are given at discrete data points, and*
- 2. use Lagrange polynomial interpolation to find derivatives of discrete functions.*

To find the derivatives of functions that are given at discrete points, several methods are available. Although these methods are mainly used when the data is spaced unequally, they can be used for data that is spaced equally as well.

Forward Difference Approximation of the First Derivative

We know

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite Δx ,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

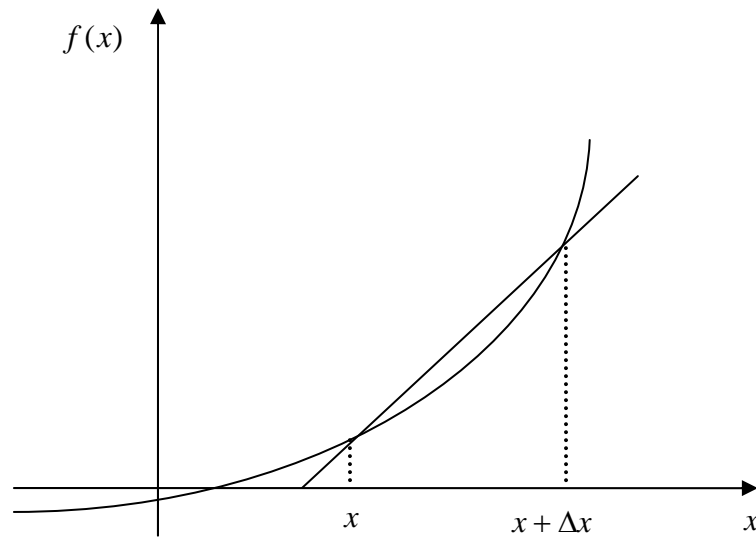


Figure 1 Graphical representation of forward difference approximation of first derivative.

So given $n+1$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, the value of $f'(x)$ for $x_i \leq x \leq x_{i+1}$, $i = 0, \dots, n-1$, is given by

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

Example 1

To find the stress concentration around a hole in a plate under a uniform stress, a finite difference program has been written that calculates the radial and tangential displacements at different points in the plate. To find the stresses and hence the stress concentration factor, one needs to find the derivatives of these displacements. In Table 1 the radial displacements u are given along the y -axis. The radius of the hole is 1.0 cm.

- At $x = 0$, if the radial strain ε_r is given by $\varepsilon_r = \frac{\partial u}{\partial r}$, find the radial strain at $r = 1.1$ cm using the forward divided difference method.
- If the tangential strain at $r = 1.1$ cm, $\theta = 90^\circ$ is given to you as $\varepsilon_\theta = 0.0029733$, find the hoop stress, σ_θ , at $r = 1.1$ cm, $\theta = 90^\circ$ if $\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$, where $E = 200$ GPa and $\nu = 0.3$.

Table 1 Radial displacement as a function of location.

r (cm)	u (cm)
1.0	-0.0010000
1.1	-0.0010689
1.2	-0.0011088
1.3	-0.0011326
1.4	-0.0011474
1.5	-0.0011574
1.6	-0.0011650
1.7	-0.0011718
1.8	-0.0011785
1.9	-0.0011857

Solution

a)

$$\begin{aligned}\varepsilon_r &= \frac{\partial u}{\partial r} \\ &\approx \frac{u_{i+1} - u_i}{\Delta r} \\ r_i &= 1.1 \\ r_{i+1} &= 1.2 \\ \Delta r &= r_{i+1} - r_i \\ &= 1.2 - 1.1 \\ &= 0.1 \\ u_{i+1} &= -0.0011088 \\ u_i &= -0.0010689 \\ \varepsilon_r &= \frac{u_{i+1} - u_i}{\Delta r} \\ &= \frac{-0.0011088 - (-0.0010689)}{0.1} \\ &= -0.000399 \text{ cm/cm}\end{aligned}$$

$$\begin{aligned}\text{b)} \quad \sigma_\theta &= \frac{E}{1-\nu^2} (\varepsilon_r + \nu\varepsilon_\theta) \\ &= \frac{2 \times 10^{11}}{1-0.3^2} (-0.000399 + 0.3 \times 0.0029733) \\ &= 108.35 \times 10^6 \text{ Pa}\end{aligned}$$

Direct Fit Polynomials

In this method, given $n + 1$ data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, one can fit a n^{th} order polynomial given by

$$P_n(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} + a_nx^n$$

To find the first derivative,

$$P'_n(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2x + \dots + (n-1)a_{n-1}x^{n-2} + na_nx^{n-1}$$

Similarly, other derivatives can also be found.

Example 2

To find the stress concentration around a hole in a plate under a uniform stress, a finite difference program has been written that calculates the radial and tangential displacements at different points in the plate. To find the stresses and hence the stress concentration factor, one needs to find the derivatives of these displacements. In Table 2 the radial displacements u are given along the y -axis. The radius of the hole is 1.0 cm.

- At $x = 0$, if the radial strain ε_r is given by $\varepsilon_r = \frac{\partial u}{\partial r}$, find the radial strain at $r = 1.1$ cm. Use a third order polynomial interpolant for calculating the radial strain.
- If the tangential strain at $r = 1.1$ cm, $\theta = 90^\circ$ is given to you as $\varepsilon_\theta = 0.0029733$, find the hoop stress, σ_θ , at $r = 1.1$ cm, $\theta = 90^\circ$ if $\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$, where $E = 200$ GPa and $\nu = 0.3$.

Table 2 Radial displacement as a function of location.

r (cm)	u (cm)
1.0	-0.0010000
1.1	-0.0010689
1.2	-0.0011088
1.3	-0.0011326
1.4	-0.0011474
1.5	-0.0011574
1.6	-0.0011650
1.7	-0.0011718
1.8	-0.0011785
1.9	-0.0011857

Solution

For third order polynomial interpolation (also called cubic interpolation), we choose the displacement given by

$$u(r) = a_0 + a_1r + a_2r^2 + a_3r^3$$

Since we want to find the radial strain at $r = 1.1$ cm, and we are using a third order polynomial, we need to choose the four points closest to $r = 1.1$ that also bracket $r = 1.1$ to evaluate it.

The four points are $r_0 = 1.0$, $r_1 = 1.1$, $r_2 = 1.2$ and $r_3 = 1.3$.

$$r_0 = 1.0, \quad u(r_0) = -0.0010000$$

$$r_1 = 1.1, \quad u(r_1) = -0.0010689$$

$$r_2 = 1.2, \quad u(r_2) = -0.0011088$$

$$r_3 = 1.3, \quad u(r_3) = -0.0011326$$

such that

$$u(1.0) = -0.0010000 = a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3$$

$$u(1.1) = -0.0010689 = a_0 + a_1(1.1) + a_2(1.1)^2 + a_3(1.1)^3$$

$$u(1.2) = -0.0011088 = a_0 + a_1(1.2) + a_2(1.2)^2 + a_3(1.2)^3$$

$$u(1.3) = -0.0011326 = a_0 + a_1(1.3) + a_2(1.3)^2 + a_3(1.3)^3$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1.1 & 1.21 & 1.331 \\ 1 & 1.2 & 1.44 & 1.728 \\ 1 & 1.3 & 1.69 & 2.197 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -0.0010000 \\ -0.0010689 \\ -0.0011088 \\ -0.0011326 \end{bmatrix}$$

Solving the above gives

$$a_0 = 0.004122$$

$$a_1 = -0.011517$$

$$a_2 = 0.008545$$

$$a_3 = -0.00215$$

Hence

$$\begin{aligned} u(r) &= a_0 + a_1 r + a_2 r^2 + a_3 r^3 \\ &= 0.004122 - 0.011517r + 0.008545r^2 - 0.00215r^3, \quad 1 \leq r \leq 1.3 \end{aligned}$$

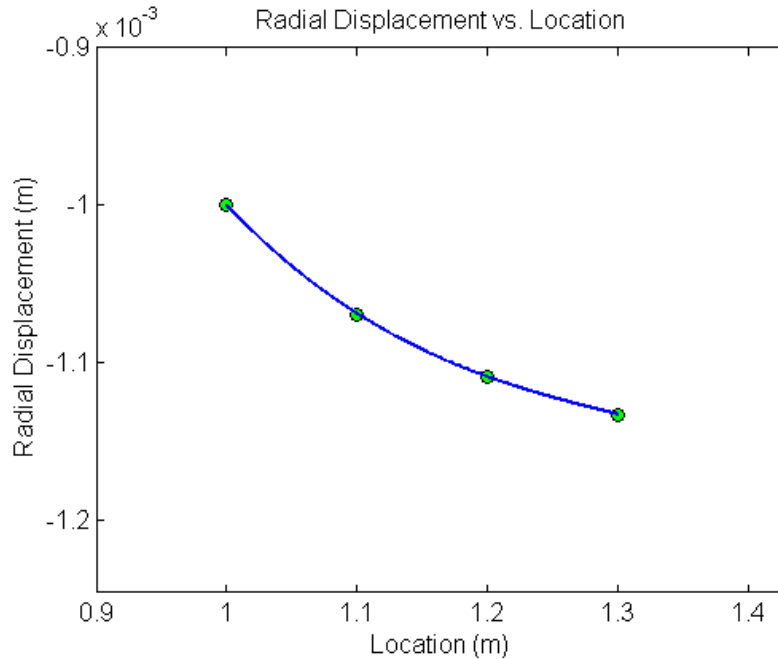


Figure 2 Graph of radial displacement vs. location.

The derivative of radial displacement at $r = 1.1$ cm is given by

$$u'(1.1) = \left. \frac{d}{dr} u(r) \right|_{r=1.1}$$

Given that $u(r) = 0.004122 - 0.011517r + 0.008545r^2 - 0.00215r^3$, $1 \leq r \leq 1.3$,

$$\begin{aligned} u'(r) &= \frac{d}{dr} u(r) \\ &= \frac{d}{dr} (0.004122 - 0.011517r + 0.008545r^2 - 0.00215r^3) \\ &= -0.011517 + 0.01709r - 0.00645r^2, \quad 1 \leq r \leq 1.3 \end{aligned}$$

$$\begin{aligned} u'(1.1) &= -0.011517 + 0.01709(1.1) - 0.00645(1.1)^2 \\ &= -0.0005225 \text{ cm/cm} \end{aligned}$$

$$\varepsilon_r = -0.00052250 \text{ cm/cm}$$

$$\begin{aligned} \text{b) } \sigma_\theta &= \frac{E}{1-\nu^2} (\varepsilon_r + \nu\varepsilon_\theta) \\ &= \frac{2 \times 10^{11}}{1-0.3^2} (-0.00052250 + 0.3 \times 0.0029733) \\ &= 81.207 \times 10^6 \text{ Pa} \end{aligned}$$

Lagrange Polynomial

In this method, given $(x_0, y_0), \dots, (x_n, y_n)$, one can fit a n^{th} order Lagrangian polynomial given by

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j=i$ omitted.

Then to find the first derivative, one can differentiate $f_n(x)$ once, and so on for other derivatives.

For example, the second order Lagrange polynomial passing through (x_0, y_0) , (x_1, y_1) , and (x_2, y_2) is

$$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Differentiating the above equation gives

$$f_2'(x) = \frac{2x - (x_1 + x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2x - (x_0 + x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2x - (x_0 + x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Differentiating again would give the second derivative as

$$f_2''(x) = \frac{2}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Example 3

To find the stress concentration around a hole in a plate under a uniform stress, a finite difference program has been written that calculates the radial and tangential displacements at different points in the plate. To find the stresses and hence the stress concentration factor, one needs to find the derivatives of these displacements. In Table 3 the radial displacements u are given along the y -axis. The radius of the hole is 1.0 cm.

- At $x = 0$ if the radial strain ε_r is given by $\varepsilon_r = \frac{\partial u}{\partial r}$, find the radial strain at $r = 1.1$ cm. Use a second order Lagrange polynomial interpolant for calculating the radial strain.
- If the tangential strain at $r = 1.1$ cm, $\theta = 90^\circ$ is given to you as $\varepsilon_\theta = 0.0029733$, find the hoop stress, σ_θ , at $r = 1.1$ cm, $\theta = 90^\circ$ if $\sigma_\theta = \frac{E}{1-\nu^2} (\varepsilon_r + \nu\varepsilon_\theta)$, where $E = 200$ GPa and $\nu = 0.3$.

Table 3 Radial displacement as a function of location.

r (cm)	u (cm)
1.0	-0.0010000
1.1	-0.0010689
1.2	-0.0011088
1.3	-0.0011326
1.4	-0.0011474
1.5	-0.0011574
1.6	-0.0011650
1.7	-0.0011718
1.8	-0.0011785
1.9	-0.0011857

Solution

For second order Lagrangian interpolation, we choose the radial displacement given by

$$u(r) = \left(\frac{r-r_1}{r_0-r_1} \right) \left(\frac{r-r_2}{r_0-r_2} \right) u(r_0) + \left(\frac{r-r_0}{r_1-r_0} \right) \left(\frac{r-r_2}{r_1-r_2} \right) u(r_1) + \left(\frac{r-r_0}{r_2-r_0} \right) \left(\frac{r-r_1}{r_2-r_1} \right) u(r_2)$$

(a) Change in the radial displacement at 1.1 cm :

Since we want to find the rate of change in the radial displacement at $r = 1.1$ cm , and we are using second order Lagrangian interpolation, we need to choose the three points closest to $r = 1.1$ cm that also bracket $r = 1.1$ cm to evaluate it.

The three points are $r_0 = 1.0$, $r_1 = 1.1$, and $r_2 = 1.2$.

$$r_0 = 1.0, \quad u(r_0) = -0.0010000$$

$$r_1 = 1.1, \quad u(r_1) = -0.0010689$$

$$r_2 = 1.2, \quad u(r_2) = -0.0011088$$

The change in the radial displacement at $r = 1.1$ cm is given by

$$\frac{du(1.1)}{dr} = \frac{d}{dr} u(r) \Big|_{r=1.1}$$

Hence

$$u'(r) = \frac{2r - (r_1 + r_2)}{(r_0 - r_1)(r_0 - r_2)} u(r_0) + \frac{2r - (r_0 + r_2)}{(r_1 - r_0)(r_1 - r_2)} u(r_1) + \frac{2r - (r_0 + r_1)}{(r_2 - r_0)(r_2 - r_1)} u(r_2)$$

$$\begin{aligned} u'(1.1) &= \frac{2(1.1) - (1.1 + 1.2)}{(1.0 - 1.1)(1.0 - 1.2)} (-0.0010000) + \frac{2(1.1) - (1.0 + 1.2)}{(1.1 - 1.0)(1.1 - 1.2)} (-0.0010689) \\ &\quad + \frac{2(1.1) - (1.0 + 1.1)}{(1.2 - 1.0)(1.2 - 1.1)} (-0.0011088) \end{aligned}$$

$$= -5(-0.0010000) + 0(-0.0010689) + 5(-0.0011088)$$

$$= -0.000544 \text{ cm/cm}$$

$$\begin{aligned} \text{b) } \sigma_{\theta} &= \frac{E}{1-\nu^2} (\varepsilon_r + \nu\varepsilon_{\theta}) \\ &= \frac{2 \times 10^{11}}{1-0.3^2} (-0.000544 + 0.3 \times 0.0029733) \\ &= 76.481 \times 10^6 \text{ Pa} \end{aligned}$$

DIFFERENTIATION

Topic	Differentiation of Discrete Functions
Summary	These are textbook notes differentiation of discrete functions
Major	Civil Engineering
Authors	Autar Kaw, Luke Snyder
Date	June 20, 2009
Web Site	Uhttp://numericalmethods.eng.usf.edu
