Chapter 03.00C

Physical Problem for Nonlinear Equations Civil Engineering

Problem

My spouse asked me to build a bookshelf (Figure 1) for her books. Her books range from $8\frac{1}{2}''$ in to $11''$ in height, and would take $29''$ of space along the length. She asked me how much would the bookshelf sag, as she does not like sagging bookshelves we had during graduate student days. After all, our tolerance to sagging does sag after graduation! So, as a true engineer, before going to the local building store, I modeled the problem to find the maximum deflection of the bookshelf. I assumed a shelf of thickness $3/8''$ and width $12''$. The Young's modulus of a typical wood shelf material is 3.667 Msi.

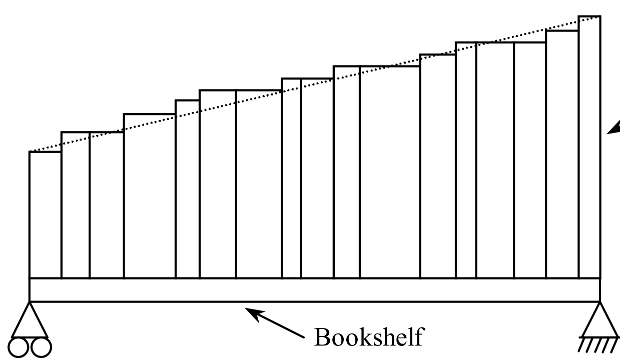

Figure 1 Distributions of books on the bookshelf.

Solution

I assume the following

- 100 pages of $8\frac{1}{2}''\times 11''$ book paper weighs 1 lb (on its website www.usps.gov, United States Post Office suggests 6 letter size pages weigh about 1 ounce, so 100 pages would be approximately a pound).

- 400 pages take up the space of 1" in thickness (I actually measured several textbooks and found an average.)
- the books decrease linearly in height from 11" from the right end to $8\frac{1}{2}$" on the left, and the book width is $8\frac{1}{2}$" for all books,

The total weight W of the books is found as follows. If all the books were of dimensions $11"\times 8\frac{1}{2}"$, the weight of the books would be

$$W = 29\,\text{in} \times 400\,\frac{\text{pages}}{\text{in}} \times \frac{1\,\text{lb}}{100\,\text{pages}}$$
$$= 116\,lb$$

But since the books decrease in height from 11" to $8\frac{1}{2}$", while the width of the books is assumed to be constant at $8\frac{1}{2}$", the actual total weight of the books is calculated as

$$W = \frac{\frac{1}{2}\left(11 + 8\frac{1}{2}\right)29}{(11)(29)} \times 116$$
$$\approx 103\,\text{lb}$$

Now assuming that the weight 'W' is distributed linearly from the left end to the right as a distribution q (Figure 2),

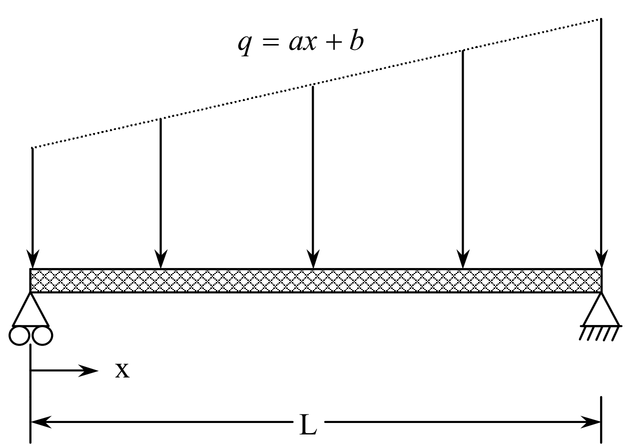

Figure 2 Approximate weight distribution of books.

Then the total weight W over the length L

$$W = \int_0^L q\,dx$$
$$= \int_0^L (ax+b)\,dx$$

$$= a\frac{L^2}{2} + bL \quad (1)$$

Also, the weight distribution can be assumed to be in the same ratio as the height of the books at the two ends, then

$$\frac{q(x=0)}{q(x=L)} = \frac{8.5}{11}$$

$$\frac{a(0)+b}{aL+b} = \frac{8.5}{11}$$

$$\frac{b}{aL+b} = \frac{8.5}{11} \quad (2)$$

Substituting

$$W = 103 \text{ lb},$$

$$L = 29'',$$

in Equations (1) and (2), we get

$$a\frac{29^2}{2} + b(29) = 103$$

$$\frac{b}{a(29)+b} = \frac{8.5}{11}$$

or

$$420.5a + 29b = 103$$

$$-246.5a + 3b = 0$$

The above set of equations gives the solution as

$$a = 0.031403$$

$$b = 3.0964$$

So the weight distribution is given by

$$q = 0.031403x + 3.0964, \quad 0 \le x \le 29.$$

Now let us find the deflection in the beam. The deflection y as a function of x along the length of the beam is given by

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

where

M = Bending moment (lb.in)

E = Young's modulus (psi)

I = Second moment of area (in^4)

To find the bending moment, we need to first find the reaction force at the support. Let R_1 and R_2 be the reactions at the left and right support, respectively (Figure 3). Then from the sum of forces in the vertical direction,

$$R_1 + R_2 = W$$

and the moment at the left support is zero as it is simply supported

$$R_2 L - \int_0^L (ax+b)x\,dx = 0$$

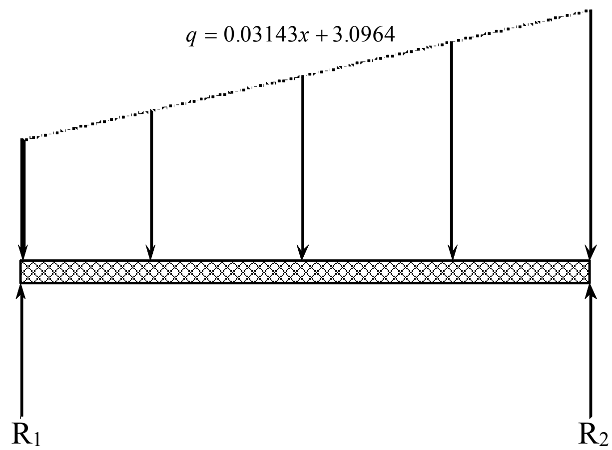

Figure 3 Reaction at the end of shelf supports.

$$R_2 L - a\frac{L^3}{3} - \frac{bL^2}{2} = 0$$

Substituting

$W = 103$,

$a = 0.031403$,

$b = 3.0964$,

$L = 29$,

the equilibrium equations are

$$R_1 + R_2 = 103$$

$$R_2(29) - (0.031403)\frac{29^3}{3} - 3.0964\frac{29^2}{2} = 0$$

gives

$$R_1 = 49.300\, lb$$

$$R_2 = 53.701\, lb$$

The bending moment at any cross-section at a distance of x from the left end is then given by summing the moments at a cross-section of distance x from the left end as (Figure 4)

$$M(x) - R_1 x + \int_0^L (ax' + b)(x - x')dx' = 0$$

$$M(x) = R_1 x \int_0^L (ax' + b)(x - x')dx'$$

$$= R_1 x - \frac{1}{6}ax^3 - \frac{1}{2}bx^2$$

Substituting value of R_1, a and b,

$R_1 = 49.299$

$a = 0.031403$

$b = 3.0964$

the bending moment equation is given by

$$M(x) = 49.299x - \frac{1}{6}(0.031403)x^3 - \frac{1}{2}(3.0964)x^2$$
$$M(x) = 49.299x - 0.0052339x^3 - 1.5482x^2$$

So

$$EI\frac{d^2v}{dx^2} = M(x)$$
$$\frac{d^2v}{dx^2} = \frac{M(x)}{EI}$$

$$E = 3.667 Msi$$
$$I = \frac{1}{12}ht^3$$

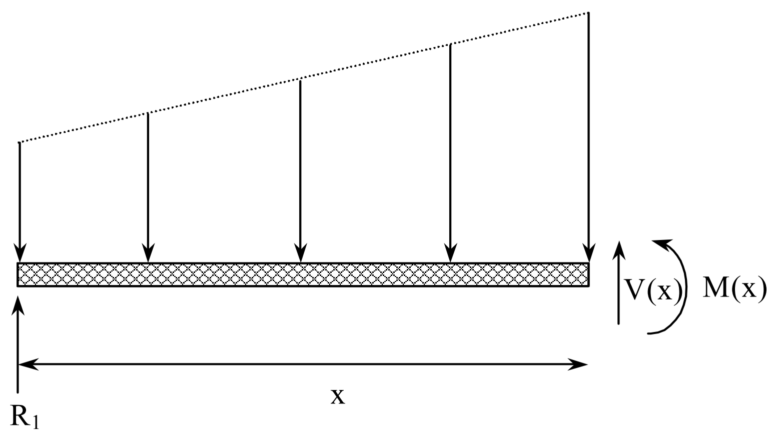

Figure 4 Free body diagram to find bending moment at a cross-section.

where

h = width of shelf
t = thickness of shelf

$$I = \frac{1}{12} \times 12 \times \left(\frac{3}{8}\right)^3$$
$$= 0.052734 \text{in}$$

Hence

$$\frac{d^2v}{dx^2} = \frac{M(x)}{EI}$$
$$= \frac{49.29x - 0.0052339x^3 - 1.5428x^2}{3.6667 \times 10^6 \times 0.052734}$$
$$= 0.25496 \times 10^{-3}x - 0.27068 \times 10^{-7}x^3 - 0.80067 \times 10^{-5}x^2$$

Integrating with respect to x gives

$$\frac{dv}{dx} = 0.25496\times10^{-3}x^2 - 0.67670\times10^{-8}x^4 - 0.26689\times10^{-5}x^3 + c_1$$

Integrating once again with respect to x gives

$$v(x) = 0.42493\times10^{-4}x^3 - 0.13534\times10^{-8}x^5 - 0.66723\times10^{-6}x^4 + c_1x + c_2$$

Now the boundary conditions at $x = 0$ and $x = \mathrm{L}$ are that the displacement is zero at the ends,

$$v(0) = 0$$
$$v(L) = 0$$

So from

$$v(0) = 0$$

we get

$c_2 = 0$, and

from

$v(L) = 0$,

we get

$$0 = 0.42493\times10^{-3}L^3 - 0.13534\times10^{-8}L^5 - 0.66723\times10^{-6}L^4 + c_1L$$

Substituting $\mathrm{L} = 29''$

$$0 = 0.42493\times10^{-3}(29)^3 - 0.13534\times10^{-8}(29)^5 - 0.66723\times10^{-6}(29)^4 + c_1(29)$$
$$c_1 = -0.018506$$

Hence the vertical deflection in the beam is given by

$$v(x) = 0.42493\times10^{-4}x^3 - 0.13534\times10^{-8}x^5 - 0.66723\times10^{-6}x^4 - 0.018506x$$

But to find where the deflection would be maximum, we need to take the first derivative of the deflection to find where

$$\frac{dv}{dx} = 0,$$

that is

$$\frac{d}{dx}(0.42493\times10^{-4}x^3 - 0.13534\times10^{-8}x^5 - 0.66723\times10^{-6}x^4 - 0.018506x) = 0$$
$$f(x) = 0.12748\times10^{-3}x^2 - 0.67670\times10^{-8}x^4 - 0.26689\times10^{-5}x^3 - 0.018506 = 0$$

Now here is a nonlinear equation that needs to solved to find where the maximum deflection occurs. Then substituting the obtained value of x in $v(x)$ would give us the maximum sagging in the beam.

Topic	NONLINEAR EQUATIONS
Sub Topic	Physical Problem
Summary	A physical problem of finding the maximum sagging in a bookshelf.
Authors	Autar Kaw
Date	December 7, 2008
Web Site	http://numericalmethods.eng.usf.edu