

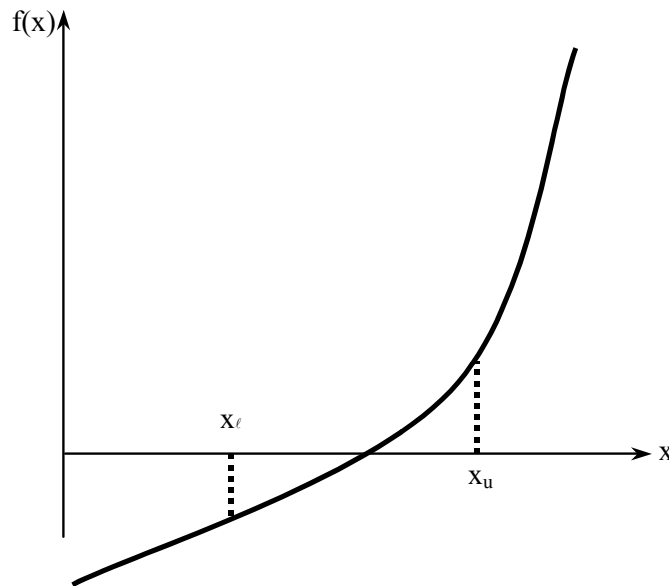
Roots of a Nonlinear Equation

Topic: Bisection Method

Major: Civil Engineering

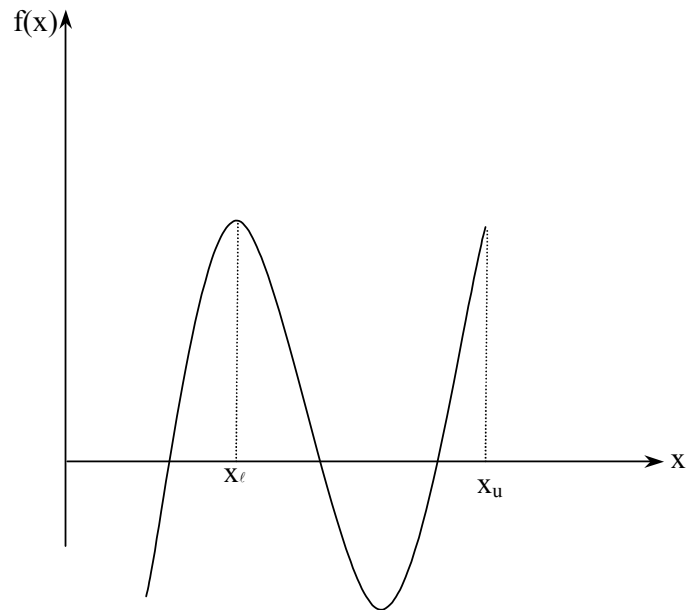
Basis of Bisection Method

Theorem: An equation $f(x)=0$, where $f(x)$ is a real continuous function, has at least one root between x_l and x_u if $f(x_l) f(x_u) < 0$.



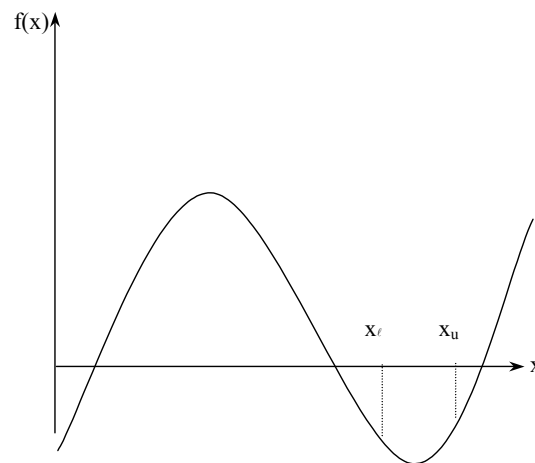
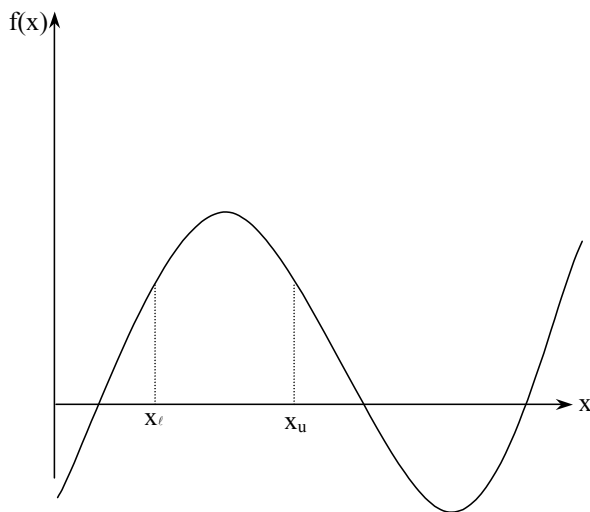
Theorem

If function $f(x)$ in $f(x)=0$ does not change sign between two points, roots may still exist between the two points.



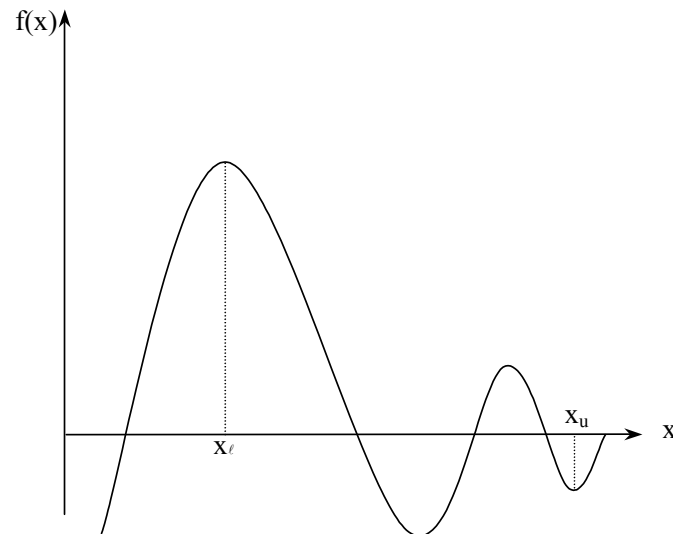
Theorem

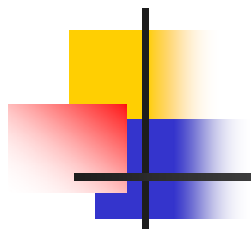
If the function $f(x)$ in $f(x)=0$ does not change sign between two points, there may not be any roots between the two points.



Theorem

If the function $f(x)$ in $f(x)=0$ changes sign between two points, more than one root may exist between the two points.

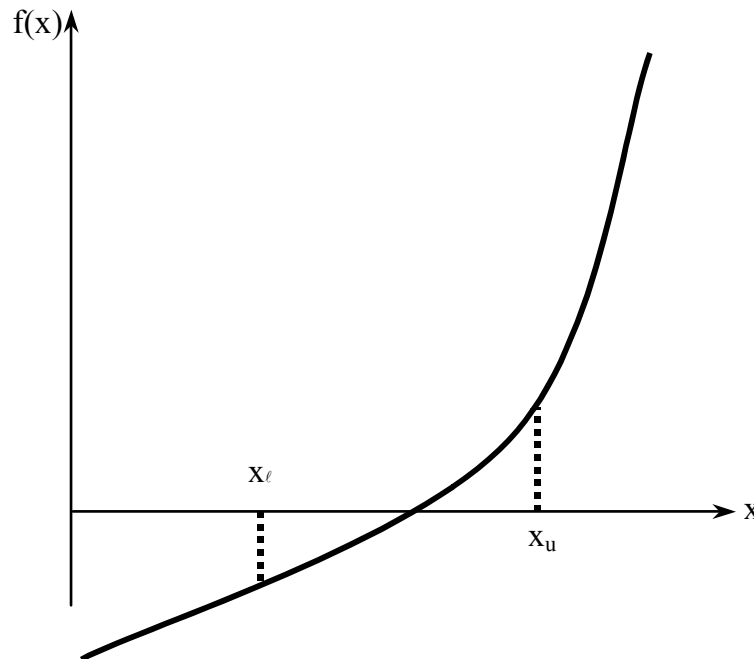




Algorithm for Bisection Method

Step 1

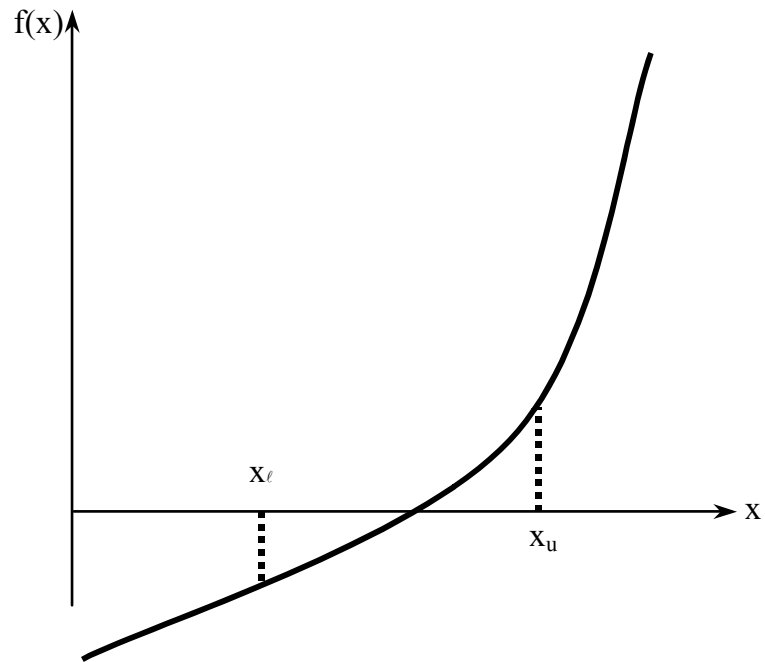
- Choose x_l and x_u as two guesses for the root such that $f(x_l) f(x_u) < 0$, or in other words, $f(x)$ changes sign between x_l and x_u .



Step 2

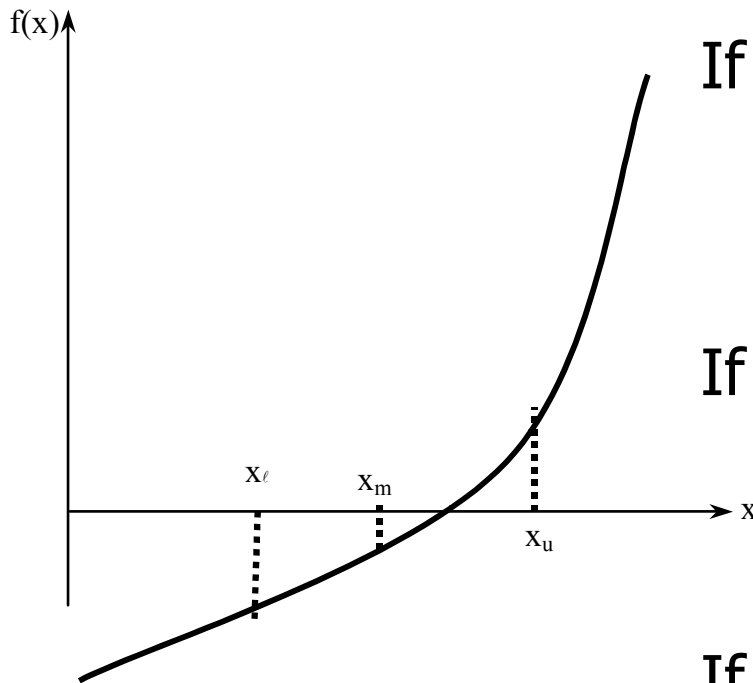
Estimate the root, x_m of the equation $f(x) = 0$ as the mid-point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$



Step 3

Now check the following



If $f(x_l) f(x_m) < 0$, then the root lies between x_R and x_m ; then $x_l = x_l$; $x_u = x_m$.

If $f(x_R) f(x_m) > 0$, then the root lies between x_m and x_u ; then $x_l = x_m$; $x_u = x_u$.

If $f(x_l) f(x_m) = 0$; then the root is x_m .
Stop the algorithm if this is true.



Step 4

New estimate

$$x_m = \frac{x_\ell + x_u}{2}$$

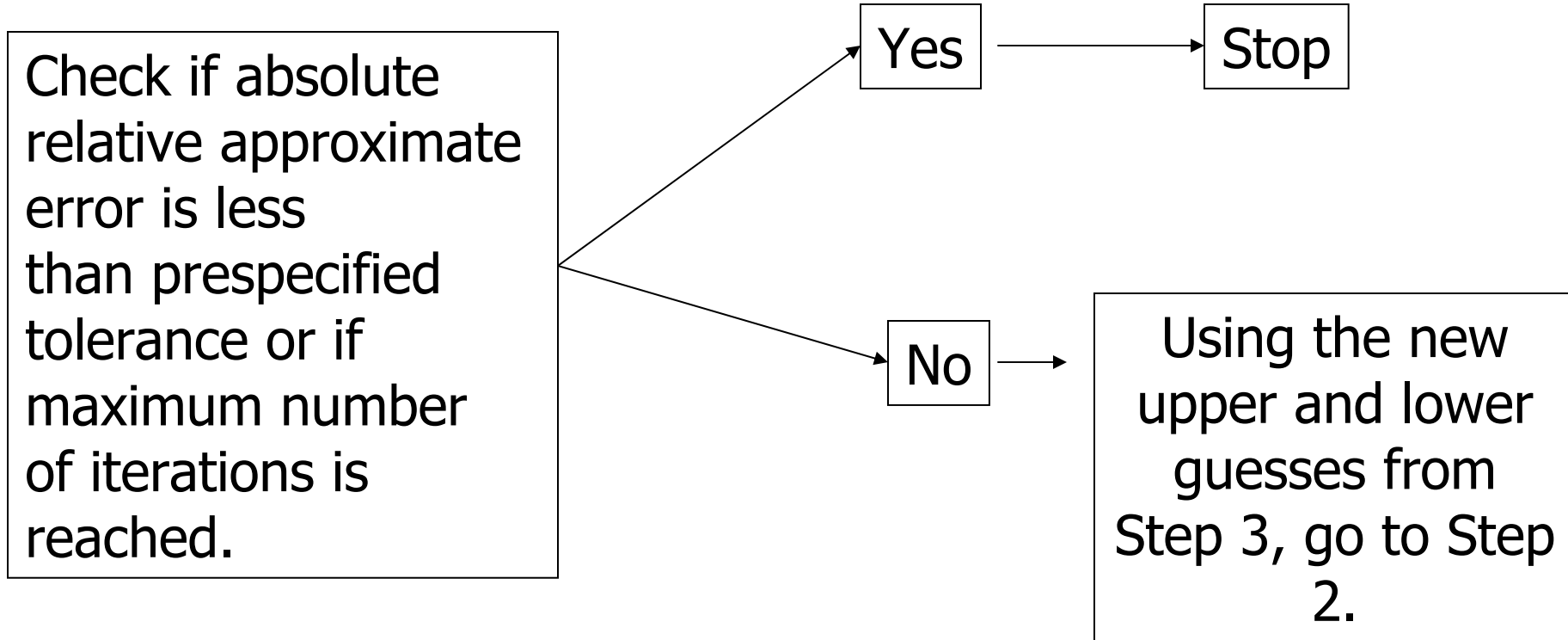
Absolute Relative Approximate Error

$$|\epsilon_a| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100$$

x_m^{old} = previous estimate of root

x_m^{new} = current estimate of root

Step 5





Example

- You are making a bookshelf to carry books that range from 8 ½ " to 11" in height and would take 29" of space along length. The material is wood having Young's Modulus 3.667 Msi, thickness 3/8 " and width 12". You are asked to find the maximum deflection of the bookshelf.

The vertical deflection of the shelf is given by

$$v(x) = 0.42493 \times 10^{-4} x^3 - 0.13533 \times 10^{-8} x^5 - 0.66722 \times 10^{-6} x^4 - 0.018507x$$

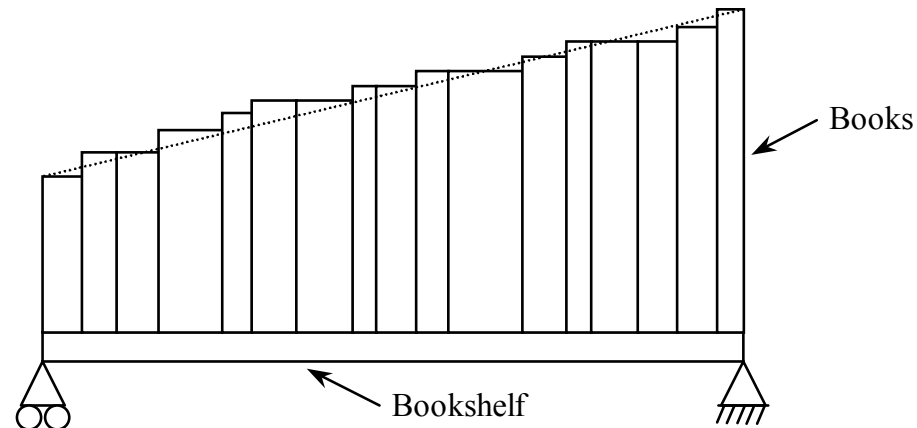
where x is the position where the deflection is maximum. Hence to find the maximum deflection we need to find where $f(x) = \frac{dv}{dx} = 0$

Solution

The equation that gives the position 'x' where the deflection is maximum is given by:

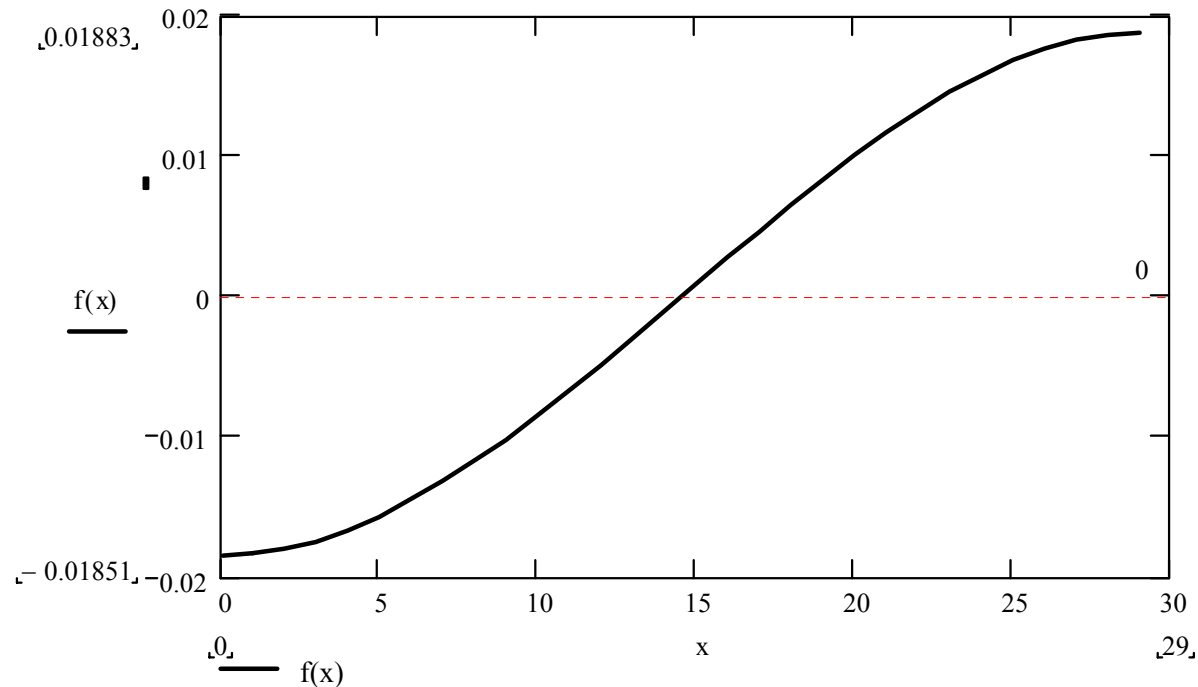
$$f(x) = -0.67665 \times 10^{-8} x^4 - 0.26689 \times 10^{-5} x^3 + 0.12748 \times 10^{-3} x^2 - 0.018507 = 0$$

Use the bisection method of finding roots of equations to find the depth 'x' to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation.

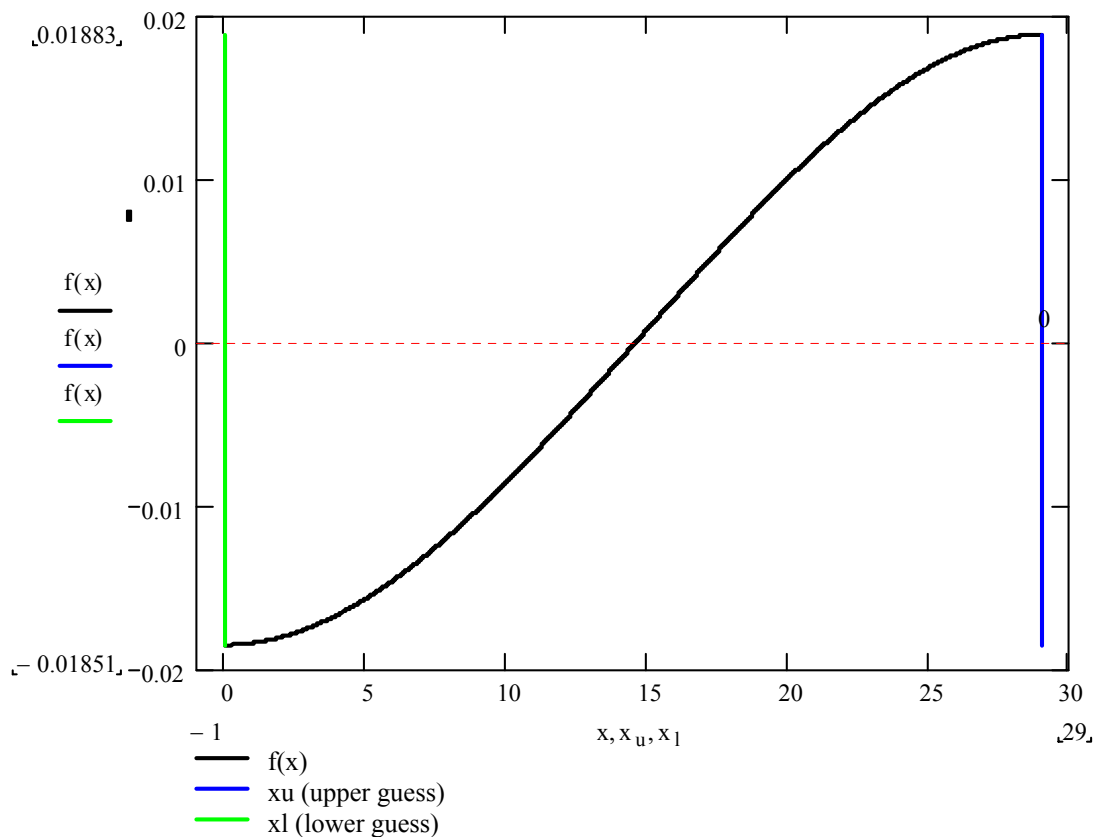


Graph of function $f(x)$

$$f(x) = -0.67665 \times 10^{-8} x^4 - 0.26689 \times 10^{-5} x^3 + 0.12748 \times 10^{-3} x^2 - 0.018507 = 0$$



Checking if the bracket is valid



Choose the bracket

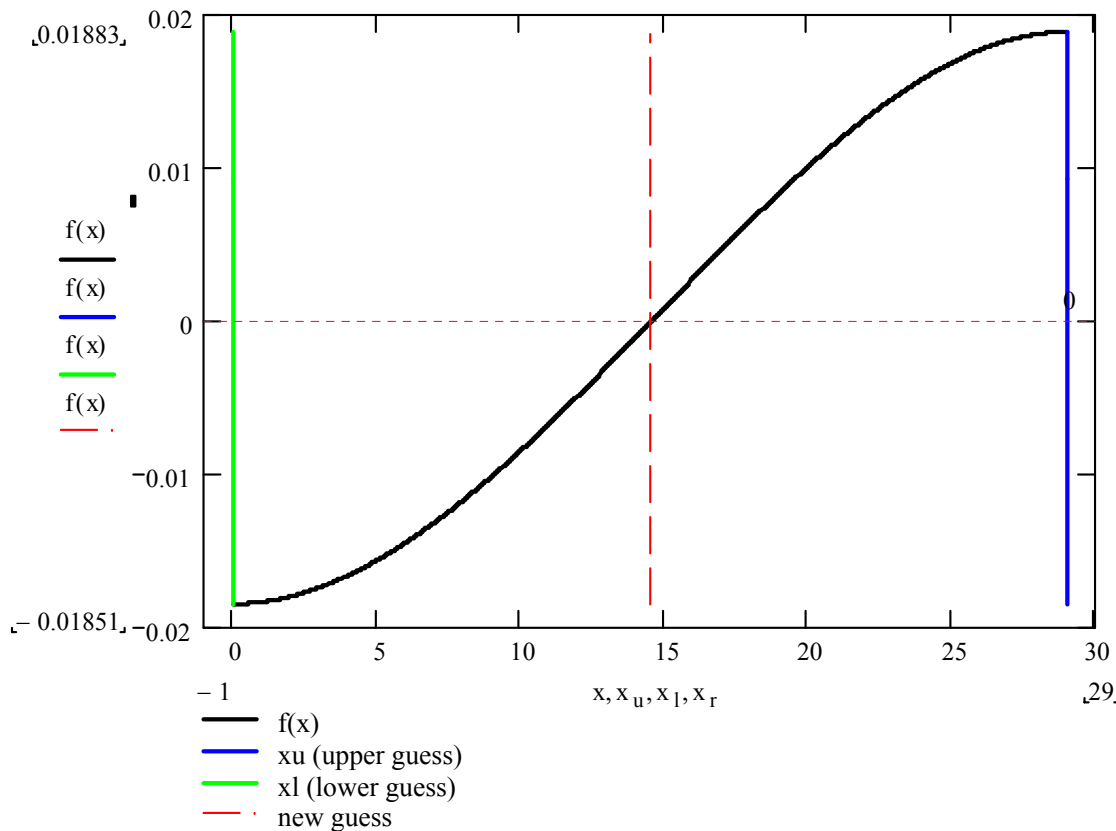
$$x_l = 0$$

$$x_u = 29$$

$$f(0) = -0.018507$$

$$f(29) = 0.01883$$

Iteration #1



$$x_\ell = 0, x_u = 29$$

$$x_m = \frac{0 + 29}{2} = 14.5$$

$$f(0) = -0.018507$$

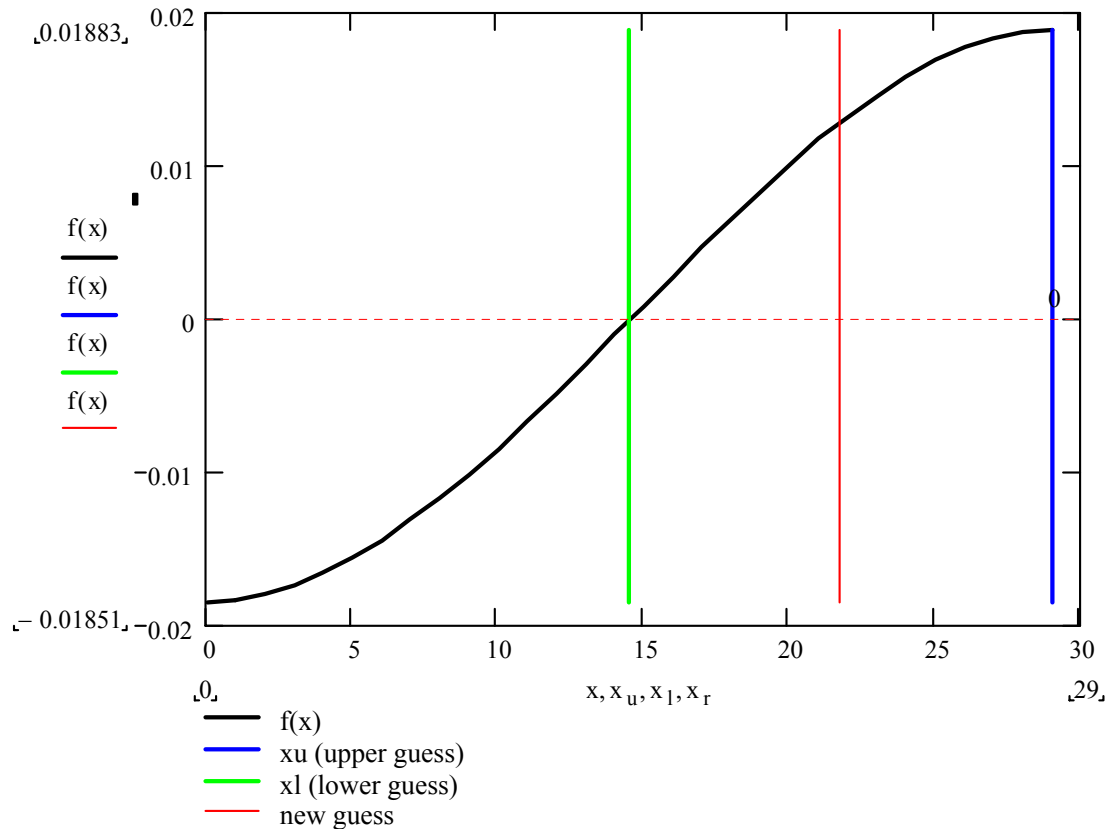
$$f(29) = 0.01883$$

$$f(14.5) = -1.3992 \times 10^{-4}$$

$$x_\ell = 14.5$$

$$x_u = 29$$

Iteration #2



$$x_l = 14.5, x_u = 29$$

$$x_m = \frac{14.5 + 29}{2} = 21.75$$

$$|\epsilon_a| = 33.33\%$$

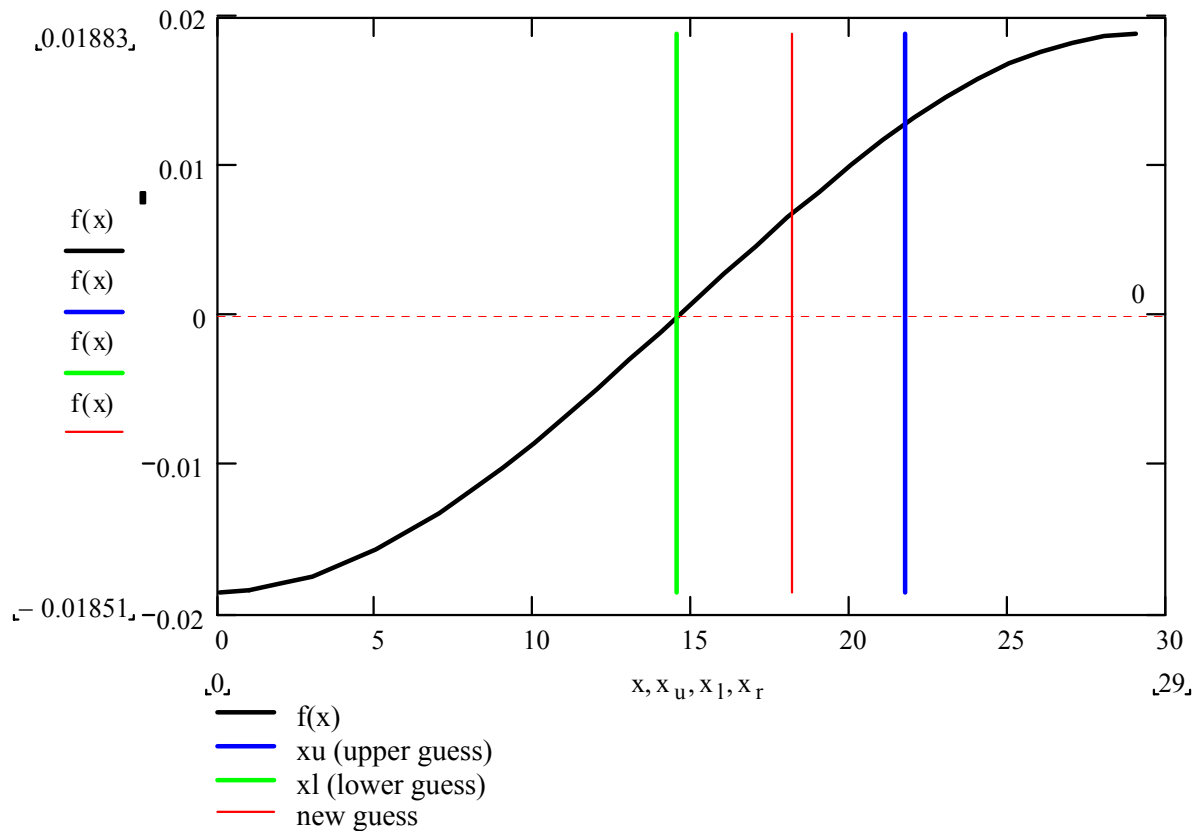
$$f(14.5) = -1.3992 \times 10^{-4}$$

$$f(29) = 0.01883$$

$$f(21.75) = 0.01282$$

$$x_l = 14.5, x_u = 21.75$$

Iteration #3



$$x_\ell = 14.5, x_u = 21.75$$

$$x_m = \frac{14.5 + 21.75}{2} = 18.125$$

$$|\epsilon_a| = 20\%$$

$$f(14.5) = -1.3992 \times 10^{-4}$$

$$f(21.5) = 0.01282$$

$$f(18.125) = 6.75036 \times 10^{-3}$$



Convergence

Table 1: Root of $f(x)=0$ as function of number of iterations for bisection method.

Iteration	x_ℓ	x_u	$ \epsilon_a $ x_m	%	$f(x_m)$
1	0	29	14.5	-----	-1.3992×10^{-4}
2	14.5	29	21.75	33.33	0.01282
3	14.5	21.75	18.125	20.00	6.7504×10^{-3}
4	14.5	18.125	16.3125	11.11	3.3512×10^{-3}
5	14.5	16.3125	15.4063	5.8824	1.6101×10^{-3}
6	14.5	15.4063	14.9531	3.0303	7.3536×10^{-4}
7	14.5	14.9531	14.7266	1.5385	2.9758×10^{-4}
8	14.5	14.7266	14.6133	0.7752	7.8859×10^{-5}
9	14.5	14.6133	14.5566	0.3891	-3.0537×10^{-5}
10	14.5566	14.6133	14.5849	0.1942	2.4159×10^{-5}



Advantages

- Always convergent
- The root bracket gets halved with each iteration - guaranteed.



Drawbacks

- Slow convergence

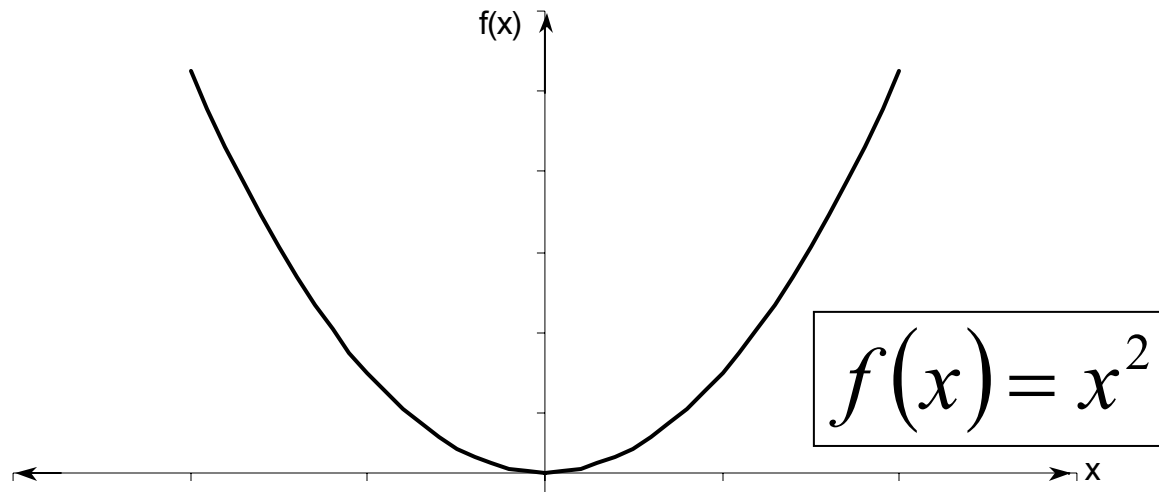


Drawbacks (continued)

- If one of the initial guesses is close to the root, the convergence is slower

Drawbacks (continued)

- If a function $f(x)$ is such that it just touches the x -axis it will be unable to find the lower and upper guesses.



Drawbacks (continued)

- Function changes sign but root does not exist

