



# Simultaneous Linear Equations

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Topic: Gaussian Elimination  
Major: Civil Engineering



# Gaussian Elimination

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One of the most popular techniques for solving simultaneous linear equations of the form  $[A][X] = [C]$

Consists of 2 steps

1. Forward Elimination of Unknowns.
2. Back Substitution



# Forward Elimination

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The goal of Forward Elimination is to transform the coefficient matrix into an Upper Triangular Matrix

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$



# Forward Elimination

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## Linear Equations

A set of  $n$  equations and  $n$  unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \quad \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n$$



# Forward Elimination

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## Transform to an Upper Triangular Matrix

Step 1: Eliminate  $x_1$  in 2<sup>nd</sup> equation using equation 1 as the pivot equation

$$\left[ \frac{\text{Eqn1}}{a_{11}} \right] \times (a_{21})$$

Which will yield

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \dots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$



# Forward Elimination

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Zeroing out the coefficient of  $x_1$  in the 2<sup>nd</sup> equation.

Subtract this equation from 2<sup>nd</sup> equation

$$\left( a_{22} - \frac{a_{21}}{a_{11}} a_{12} \right) x_2 + \dots + \left( a_{2n} - \frac{a_{21}}{a_{11}} a_{1n} \right) x_n = b_2 - \frac{a_{21}}{a_{11}} b_1$$

Or

$$a'_{22} x_2 + \dots + a'_{2n} x_n = b'_2$$

Where

$$a'_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$$

⋮

$$a'_{2n} = a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}$$



# Forward Elimination

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Repeat this procedure for the remaining equations to reduce the set of equations as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n = b'_2$$

$$a'_{32}x_2 + a'_{33}x_3 + \dots + a'_{3n}x_n = b'_3$$

$$\vdots \quad \vdots \quad \vdots$$

$$a'_{n2}x_2 + a'_{n3}x_3 + \dots + a'_{nn}x_n = b'_n$$



# Forward Elimination

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Step 2: Eliminate  $x_2$  in the 3<sup>rd</sup> equation.

Equivalent to eliminating  $x_1$  in the 2<sup>nd</sup> equation using equation 2 as the pivot equation.

$$Eqn3 - \left[ \frac{Eqn2}{a_{22}} \right] \times (a_{32})$$



# Forward Elimination

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This procedure is repeated for the remaining equations to reduce the set of equations as

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n &= b'_2 \\a''_{33}x_3 + \dots + a''_{3n}x_n &= b''_3 \\&\vdots \\a''_{n3}x_3 + \dots + a''_{nn}x_n &= b''_n\end{aligned}$$



# Forward Elimination

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Continue this procedure by using the third equation as the pivot equation and so on.

At the end of (n-1) Forward Elimination steps, the system of equations will look like:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \dots + a''_{nn}x_n = b''_3$$

$$\vdots$$

$$a^{(n-1)}_{nn}x_n = b^{(n-1)}_n$$

# Forward Elimination

At the end of the Forward Elimination steps

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ & & a''_{33} & \cdots & a''_{3n} \\ & & & \vdots & \vdots \\ & & & & a^{(n-1)}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b^{(n-1)}_n \end{bmatrix}$$



# Back Substitution

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The goal of Back Substitution is to solve each of the equations using the upper triangular matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Example of a system of 3 equations



# Back Substitution

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Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Solve the second from last equation  $(n-1)^{\text{th}}$  using  $x_n$  solved for previously.

This solves for  $x_{n-1}$ .



# Back Substitution

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Representing Back Substitution for all equations by formula

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^n a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \quad \text{For } i=n-1, n-2, \dots, 1$$

and

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

# Example: Cylinder Stresses

To find the maximum stresses in a compounded cylinder, the following four simultaneous linear equations need to be solved.

$$\begin{bmatrix} 4.2857 \times 10^7 & -9.2307 \times 10^5 & 0 & 0 \\ 4.2857 \times 10^7 & -5.4619 \times 10^5 & -4.2857 \times 10^7 & 5.4619 \times 10^5 \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857 \times 10^7 & -3.6057 \times 10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887 \times 10^3 \\ 0 \\ 0.007 \\ 0 \end{bmatrix}$$

# Example: Cylinder Stresses

If in the compounded cylinder, the inner cylinder has an internal radius of  $a=5''$ , and outer radius  $c=6.5''$ , while the outer cylinder has an internal radius of  $c=6.5''$  and outer radius,  $b=8''$ . Given  $E=30.0 \times 10^6$  psi,  $\nu=0.3$ , and that the hoop stress in outer cylinder is given by:

$$\sigma_{\theta}^2 = \frac{E}{1-\nu^2} \left[ c_3(1+\nu) + c_4 \left( \frac{1-\nu}{r^2} \right) \right]$$

find the stress on the inside radius of the outer cylinder

Find the values of  $c_1$ ,  $c_2$ ,  $c_3$  and  $c_4$  using Naïve Gauss Elimination

# Example: Cylinder Stresses

Forward Elimination: Step 1

$$\text{Row 2} - \left[ \frac{\text{Row 1}}{4.2857 \times 10^7} \right] \times (4.2857 \times 10^7) =$$

Yields

$$\begin{bmatrix} 4.2857 \times 10^7 & -9.2307 \times 10^5 & 0 & 0 \\ 0 & 3.7688 \times 10^5 & -4.2857 \times 10^7 & 5.4619 \times 10^5 \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857 \times 10^7 & -3.6057 \times 10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887 \times 10^3 \\ 7.887 \times 10^3 \\ 0.007 \\ 0 \end{bmatrix}$$

# Example: Cylinder Stresses

Forward Elimination: Step 1

$$\text{Row3} - \left[ \frac{\text{Row1}}{4.2857 \times 10^7} \right] \times (-6.5) =$$

Yields

$$\begin{bmatrix} 4.2857 \times 10^7 & -9.2307 \times 10^5 & 0 & 0 \\ 0 & 3.7688 \times 10^5 & -4.2857 \times 10^7 & 5.4619 \times 10^5 \\ 0 & -0.293839 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857 \times 10^7 & -3.6057 \times 10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887 \times 10^3 \\ 7.887 \times 10^3 \\ 5.8038 \times 10^{-3} \\ 0 \end{bmatrix}$$

# Example: Cylinder Stresses

Forward Elimination: Step 1

$$\text{Row 4} - \left[ \frac{\text{Row 1}}{4.2857 \times 10^7} \right] \times (0) =$$

Yields

$$\begin{bmatrix} 4.2857 \times 10^7 & -9.2307 \times 10^5 & 0 & 0 \\ 0 & 3.7688 \times 10^5 & -4.2857 \times 10^7 & 5.4619 \times 10^5 \\ 0 & -0.293839 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857 \times 10^7 & -3.6057 \times 10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887 \times 10^3 \\ 7.887 \times 10^3 \\ 5.8038 \times 10^{-3} \\ 0 \end{bmatrix}$$

# Example: Cylinder Stresses

Forward Elimination: Step 2

$$\text{Row 3} - \left[ \frac{\text{Row 2}}{3.7688 \times 10^5} \right] \times (-0.293839) =$$

Yields

$$\begin{bmatrix} 4.2857 \times 10^7 & -9.2307 \times 10^5 & 0 & 0 \\ 0 & 3.7688 \times 10^5 & -4.2857 \times 10^7 & 5.4619 \times 10^5 \\ 0 & 0 & -26.9140 & 0.579684 \\ 0 & 0 & 4.2857 \times 10^7 & -3.6057 \times 10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887 \times 10^3 \\ 7.887 \times 10^3 \\ 1.19530 \times 10^{-2} \\ 0 \end{bmatrix}$$

# Example: Cylinder Stresses

Forward Elimination: Step 2

$$\text{Row 4} - \left[ \frac{\text{Row 2}}{3.7688 \times 10^5} \right] \times (0) =$$

Yields

$$\begin{bmatrix} 4.2857 \times 10^7 & -9.2307 \times 10^5 & 0 & 0 \\ 0 & 3.7688 \times 10^5 & -4.2857 \times 10^7 & 5.4619 \times 10^5 \\ 0 & 0 & -26.9140 & 0.579684 \\ 0 & 0 & 4.2857 \times 10^7 & -3.6057 \times 10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887 \times 10^3 \\ 7.887 \times 10^3 \\ 1.19530 \times 10^{-2} \\ 0 \end{bmatrix}$$

# Example: Cylinder Stresses

Forward Elimination: Step 3

$$\text{Row 4} - \left[ \frac{\text{Row 3}}{-26.9140} \right] \times (4.2857 \times 10^7) =$$

Yields

$$\begin{bmatrix} 4.2857 \times 10^7 & -9.2307 \times 10^5 & 0 & 0 \\ 0 & 3.7688 \times 10^5 & -4.2857 \times 10^7 & 5.4619 \times 10^5 \\ 0 & 0 & -26.9140 & 0.579684 \\ 0 & 0 & 0 & 5.62500 \times 10^5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887 \times 10^3 \\ 7.887 \times 10^3 \\ 1.19530 \times 10^{-2} \\ 1.90336 \times 10^4 \end{bmatrix}$$

This is now ready for Back Substitution



# Example: Cylinder Stresses

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Back Substitution: Solve for  $c_4$  using the third equation

$$5.62500 \times 10^5 c_4 = 1.90336 \times 10^4$$

$$c_4 = \frac{1.90336 \times 10^4}{5.62500 \times 10^5}$$

$$= 3.38375 \times 10^{-2}$$



# Example: Cylinder Stresses

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Back Substitution: Solve for  $c_3$  using the second equation

$$-26.9140c_3 + (0.579684)c_4 = 1.19530 \times 10^{-2}$$

$$c_3 = \frac{1.19530 \times 10^{-2} - (0.579684)c_4}{-26.9140}$$

$$= \frac{1.19530 \times 10^{-2} - (0.579684) \times 3.38375 \times 10^{-2}}{-26.9140}$$

$$= 2.84687 \times 10^{-4}$$



# Example: Cylinder Stresses

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Back Substitution: Solve for  $c_2$  using the first equation

$$3.7688 \times 10^5 c_2 + (-4.2857 \times 10^7) c_3 + 5.4619 \times 10^5 c_4 = 7.887 \times 10^3$$

$$c_2 = \frac{7.887 \times 10^3 - (-4.2857 \times 10^7) c_3 - 5.4619 \times 10^5 c_4}{3.7688 \times 10^5}$$

$$= \frac{7.887 \times 10^3 - (-4.2857 \times 10^7) \times (2.84687 \times 10^{-4}) - (5.4619 \times 10^5) \times (3.38375 \times 10^{-2})}{3.7688 \times 10^5}$$

$$= 4.26164 \times 10^{-3}$$



# Example: Cylinder Stresses

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Back Substitution: Solve for  $c_1$  using the first equation

$$4.2857 \times 10^7 c_1 + -9.2307 \times 10^5 c_2 + (0)c_3 + (0)c_4 = -7.887 \times 10^3$$

$$c_1 = \frac{-7.887 \times 10^3 - (-9.2307 \times 10^5)c_2 - (0)c_3 - (0)c_4}{4.2857 \times 10^7}$$

$$= \frac{-7.887 \times 10^3 - (-9.2307 \times 10^5) \times (4.26164 \times 10^{-3})}{4.2857 \times 10^7}$$

$$= -9.22418 \times 10^{-5}$$



# Example: Cylinder Stresses

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Solution:

The solution vector is

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -9.22418 \times 10^{-5} \\ 4.26164 \times 10^{-3} \\ 2.84687 \times 10^{-4} \\ 3.38375 \times 10^{-2} \end{bmatrix}$$



# Pitfalls

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## Two Potential Pitfalls

-Division by zero: May occur in the forward elimination steps. Consider the set of equations:

$$10x_2 - 7x_3 = 7$$

$$6x_1 + 2.099x_2 + 3x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

- Round-off error: Prone to round-off errors.



# Pitfalls: Example

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Consider the system of equations:

Use five significant figures with chopping

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

At the end of Forward Elimination

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$



# Pitfalls: Example

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## Back Substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$

$$x_3 = \frac{15004}{15005} = 0.99993$$

$$x_2 = \frac{6.001 - 6x_3}{-0.001} = -1.5$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = -0.3500$$



# Pitfalls: Example

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Compare the calculated values with the exact solution

$$[X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -1.5 \\ 0.99993 \end{bmatrix}$$



# Improvements

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## Increase the number of significant digits

Decreases round off error

Does not avoid division by zero

## Gaussian Elimination with Partial Pivoting

Avoids division by zero

Reduces round off error



# Partial Pivoting

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Gaussian Elimination with partial pivoting applies row switching to normal Gaussian Elimination.

How?

At the beginning of the  $k^{\text{th}}$  step of forward elimination, find the maximum of

$$|a_{kk}|, |a_{k+1,k}|, \dots, |a_{nk}|$$

If the maximum of the values is  $|a_{pk}|$  in the  $p^{\text{th}}$  row,  $k \leq p \leq n$ ,

then switch rows  $p$  and  $k$ .



# Partial Pivoting

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What does it Mean?

Gaussian Elimination with Partial Pivoting ensures that each step of Forward Elimination is performed with the pivoting element  $|a_{kk}|$  having the largest absolute value.



# Partial Pivoting: Example

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Consider the system of equations

$$10x_1 - 7x_2 = 7$$

$$-3x_1 + 2.099x_2 + 3x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

In matrix form

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

# Partial Pivoting: Example

Forward Elimination: Step 1

Examining the values of the first column

$|10|$ ,  $|-3|$ , and  $|5|$  or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we switch row1 with row1.

Performing Forward Elimination

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

# Partial Pivoting: Example

Forward Elimination: Step 2

Examining the values of the first column

$|-0.001|$  and  $|2.5|$  or  $0.0001$  and  $2.5$

The largest absolute value is  $2.5$ , so row 2 is switched with row 3

Performing the row swap

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$



# Partial Pivoting: Example

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Forward Elimination: Step 2

Performing the Forward Elimination results in:

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$



# Partial Pivoting: Example

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Back Substitution

Solving the equations through back substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

$$x_3 = \frac{6.002}{6.002} = 1$$

$$x_2 = \frac{2.5 - 5x_3}{2.5} = 1$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = 0$$



# Partial Pivoting: Example

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Compare the calculated and exact solution

The fact that they are equal is coincidence, but it does illustrate the advantage of Partial Pivoting

$$[X]_{\text{calculated}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad [X]_{\text{exact}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$



# Summary

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- Forward Elimination
- Back Substitution
- Pitfalls
- Improvements
- Partial Pivoting