Simultaneous Linear Equations

Topic: Gauss-Seidel Method

Major: Civil Engineering

Gauss-Seidel Method

An *iterative* method.

Basic Procedure:

-Algebraically solve each linear equation for x_i

-Assume an initial guess solution array

-Solve for each x_i and repeat

-Use absolute relative approximate error after each iteration to check if error is within a pre-specified tolerance.

Gauss-Seidel Method

Why?

The Gauss-Seidel Method allows the user to control round-off error.

Elimination methods such as Gaussian Elimination and LU Decomposition are prone to prone to round-off error.

Also: If the physics of the problem are understood, a close initial guess can be made, decreasing the number of iterations needed.

Gauss-Seidel Method

Algorithm

A set of *n* equations and *n* unknowns:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

If: the diagonal elements are non-zero

Rewrite each equation solving for the corresponding unknown

ex:

First equation, solve for x_1

Second equation, solve for x_2

Gauss-Seidel Method

Algorithm

Rewriting each equation

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$ ← From Equation 1

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$ ← From equation 2

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$ ← From equation n-1

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$ ← From equation n

Gauss-Seidel Method

Algorithm

General Form of each equation

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j \neq 1}}^{n} a_{1j} x_j}{a_{11}}$$

$$x_{n-1} = \frac{c_{n-1} - \sum_{\substack{j=1 \\ j \neq n-1}}^{n} a_{n-1,j} x_j}{a_{n-1,n-1}}$$

$$x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j \neq 2}}^{n} a_{2j} x_j}{a_{22}}$$

$$x_n = \frac{c_n - \sum_{\substack{j=1 \\ j \neq n}}^{n} a_{nj} x_j}{a_{nn}}$$

Gauss-Seidel Method

Algorithm

General Form for any row 'i'

$$x_i = \frac{c_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}}, i = 1, 2, \ldots, n.$$

How or where can this equation be used?

Gauss-Seidel Method

Solve for the unknowns

Assume an initial guess for [X]

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Use rewritten equations to solve for each value of xi.

Important: Remember to use the most recent value of x_i. Which means to apply values calculated to the calculations remaining in the **current** iteration.

Gauss-Seidel Method

Calculate the Absolute Relative Approximate Error

$$\left|\varepsilon_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

So when has the answer been found?

The iterations are stopped when the absolute relative approximate error is less than a pre-specified tolerance for all unknowns.

Example: Cylinder Stresses

To find the maximum stresses in a compounded cylinder, the following four simultaneous linear equations need to solved.

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 4.2857\times10^{7} & -5.4619\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 0 \\ 0.007 \\ 0 \end{bmatrix}$$

Example: Cylinder Stresses

The system of equations is:

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 4.2857\times10^{7} & -5.4619\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 0 \\ 0.007 \\ 0 \end{bmatrix}$$

Initial Guess: Assume an initial guess of

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -0.005 \\ 0.001 \\ 0.0002 \\ 0.03 \end{bmatrix}$$

Example: Cylinder Stresses

Rewriting each equation

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 4.2857\times10^{7} & -5.4619\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 0 \\ 0.007 \\ 0 \end{bmatrix}$$

$$c_1 = \frac{-7.887\times10^{3} - (-9.2307\times10^{5})c_2 - 0c_3 - 0c_4}{4.2857\times10^{7}}$$

$$c_2 = \frac{0 - 4.2857\times10^{7}c_1 - (-4.2857\times10^{7})c_3 - 5.4619\times10^{5}c_4}{-5.4619\times10^{5}}$$

$$c_3 = \frac{0.007 - (-6.5)c_1 - (-0.15384)c_2 - 0.15384c_4}{6.5}$$

$$c_4 = \frac{0 - 0c_1 - 0c_2 - 4.2857\times10^{7}c_3}{-3.6057\times10^{5}}$$

Example: Cylinder Stresses

Substituting initial guesses into the equations

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -0.005 \\ 0.001 \\ 0.0002 \\ 0.03 \end{bmatrix}$$

$$c_1 = \frac{7.887\times10^{3} \quad (\quad 9.2307\times10^{5})\times0.001}{4.2857\times10^{7}} \quad = \quad 1.6249\times10^{\quad 4}$$

$$c_2 = \frac{0 \quad 4.2857\times10^{7}\times(\quad 1.6249\times10^{\quad 4}) \quad (\quad 4.2857\times10^{7})\times0.002 \quad 5.4619\times10^{5}\times0.03}{5.4619\times10^{5}} = 1.5569\times10^{\quad 3}$$

$$c_3 = \frac{0.007 \quad (\quad 6.5)\times(\quad 1.6249\times10^{\quad 4}) \quad (\quad 0.15384)\times1.5569\times10^{\quad 3} \quad 0.15384\times0.03}{6.5} = 2.4125\times10^{\quad 4}$$

$$c_4 = \frac{0 \quad 4.2857\times10^{7}\times2.4125\times10^{\quad 4}}{3.6057\times10^{5}} = 2.8675\times10^{\quad 2}$$

Example: Cylinder Stresses

Finding the absolute relative approximate error

$$\left|\varepsilon_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

$$\left|\varepsilon_a\right|_1 = \left|\frac{1.6249\times10^{4} \quad (\ 0.005)}{1.6249\times10^{4}}\right| x100 = 7.902\%$$

$$\left|\varepsilon_a\right|_2 = \left|\frac{1.5569\times10^{3} \quad 0.001}{1.5569\times10^{3}}\right| x100 = 175.435\%$$

$$\left|\varepsilon_a\right|_3 = \left|\frac{2.4125\times10^{4} \quad 0.002}{2.4125\times10^{4}}\right| x100 = 21.2811\%$$

$$\left|\varepsilon_a\right|_4 = \left|\frac{2.8675\times10^{2} \quad 0.03}{2.8675\times10^{2}}\right| x100 = 21.2811\%$$

At the end of the first iteration

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -1.6249\times10^{-4} \\ 1.5569\times10^{-3} \\ 2.4125\times10^{-4} \\ 2.8675\times10^{-2} \end{bmatrix}$$

The maximum absolute relative approximate error is 100.00%

Example: Cylinder Stresses

Iteration #2

Using $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -1.6249\times10^{-4} \\ 1.5569\times10^{-3} \\ 2.4125\times10^{-4} \\ 2.8675\times10^{-2} \end{bmatrix}$

$$c_1 = \frac{-7.887\times10^{3} - (-9.2307\times10^{5})\times1.559\times10^{-3}}{4.2857\times10^{7}}$$

$$= -1.5050\times10^{-4}$$

$$c_2 = \frac{0 - 4.2857\times10^{7}\times(-1.5050\times10^{-4}) - (-4.2857\times10^{7})\times2.4125\times10^{-4} - 5.4619\times10^{5}\times2.8675\times10^{-2}}{-5.4619\times10^{5}}$$

$$= -2.0639\times10^{-3}$$

$$c_3 = \frac{0.007 - (-6.5)\times(-1.5050\times10^{-4}) - (-0.15384)\times -2.0639\times10^{-3} - 0.15384\times2.8675\times10^{-2}}{6.5}$$

$$= 1.9892\times10^{-4}$$

$$c_4 = \frac{0 - 4.2857\times10^{7}\times1.9892\times10^{-4}}{-3.6057\times10^{5}} \quad = 2.3643\times10^{-2}$$

Example: Cylinder Stresses

Finding the absolute relative approximate error for the second iteration

$$\left|\varepsilon_a\right|_1 = \left|\frac{1.5050\times 10^{4} \quad \left(1.6249\times 10^{4}\right)}{1.5050\times 10^{4}}\right| x100 = 34.132\%$$

$$\left|\varepsilon_a\right|_2 = \left|\frac{2.0639\times 10^{3} \quad 1.5569\times 10^{3}}{2.0639\times 10^{3}}\right| x100 = 79.138\%$$

$$\left|\varepsilon_a\right|_3 = \left|\frac{1.9892\times 10^{4} \quad 2.4125\times 10^{4}}{1.9892\times 10^{4}}\right| x100 = 263.546\%$$

$$\left|\varepsilon_a\right|_4 = \left|\frac{2.3643\times 10^{2} \quad 2.8675\times 10^{2}}{2.3643\times 10^{2}}\right| x100 = 263.546\%$$

Example: Cylinder Stresses

At the end of the second iteration

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -1.5050\times10^{-4} \\ -2.0639\times10^{-3} \\ 1.9892\times10^{-4} \\ 2.3643\times10^{-2} \end{bmatrix}$$

The maximum absolute relative approximate error is 263.546%

Example: Cylinder Stresses

Repeating more iterations, the following values are obtained

	c_1	$\left\|\in_a\right\|_1 \%$	c_2	$\left\|\in_a\right\|_2 \%$	c_3	$\left\|\in_a\right\|_3 \%$	c_4	$\left\|\in_a\right\|_4 \%$
1	$-1.6249\text{x}10^{-4}$	7.970	$1.5569\text{x}10^{-3}$	175.435	$2.4125\text{x}10^{-4}$	21.281	$2.8675\text{x}10^{-2}$	21.281
2	$-1.5050\text{x}10^{-4}$	34.132	$-2.0639\text{x}10^{-3}$	79.138	$1.9892\text{x}10^{-4}$	263.546	$2.3643\text{x}10^{-2}$	263.546
3	$-2.2848\text{x}10^{-4}$	42.464	$-9.8931\text{x}10^{-3}$	65.826	$5.4716\text{x}10^{-5}$	134.353	$6.5035\text{x}10^{-3}$	134.353
4	$-3.9711\text{x}10^{-4}$	50.825	$-2.8949\text{x}10^{-2}$	58.524	$-1.5927\text{x}10^{-4}$	82.957	$-1.8931\text{x}10^{-2}$	82.957
5	$-8.0755\text{x}10^{-4}$	52.142	$-6.9799\text{x}10^{-2}$	58.978	$-9.3454\text{x}10^{-4}$	53.472	$-1.1108\text{x}10^{-1}$	53.472
6	$-1.6874\text{x}10^{-3}$	56.158	$-1.7015\text{x}10^{-1}$	55.589	$-2.0085\text{x}10^{-3}$	67.549	$-2.3873\text{x}10^{-1}$	67.549

Notice: The absolute relative approximate errors are not decreasing

Gauss-Seidel Method: Pitfall

What went wrong?

Even though done correctly, the answer is not converging to the correct answer

This example illustrates a pitfall of the Gauss-Seidel method: not all systems of equations will converge.

Is there a fix?

One class of system of equations always converges: One with a *diagonally dominant* coefficient matrix.

Diagonally dominant: [A] in [A] [X] = [C] is diagonally dominant if:

$$\left|a_{\mathrm{ii}}\right| \geq \sum_{\substack{j=1 \\ j\neq \mathrm{i}}}^{n} \left|a_{ij}\right|$$ for all 'i' and $$\left|a_{ii}\right| \rangle \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right|$$ for at least one 'i'

Gauss-Seidel Method: Pitfall

Diagonally Dominant: In other words….

For every row: the element on the diagonal needs to be equal than or greater than the sum of the other elements of the coefficient matrix

For at least one row: The element on the diagonal needs to be greater than the sum of the elements.

What can be done? If the coefficient matrix is not originally diagonally dominant, the rows can be rearranged to make it diagonally dominant.

Example: Cylinder Stresses

Examination of the coefficient matrix reveals that it is not diagonally dominant and cannot be rearranged to become diagonally dominant

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 4.2857\times10^{7} & -5.4619\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix}$$

This particular problem is an example of a system of linear equations that cannot be solved using the Gauss-Seidel method.

Other methods that would work:

1. Gaussian elimination

2. LU Decomposition

Gauss-Seidel Method

Summary

-Advantages of the Gauss-Seidel Method

-Algorithm for the Gauss-Seidel Method

-Pitfalls of the Gauss-Seidel Method

Gauss-Seidel Method

Questions?