Holistic Numerical Methods Institute

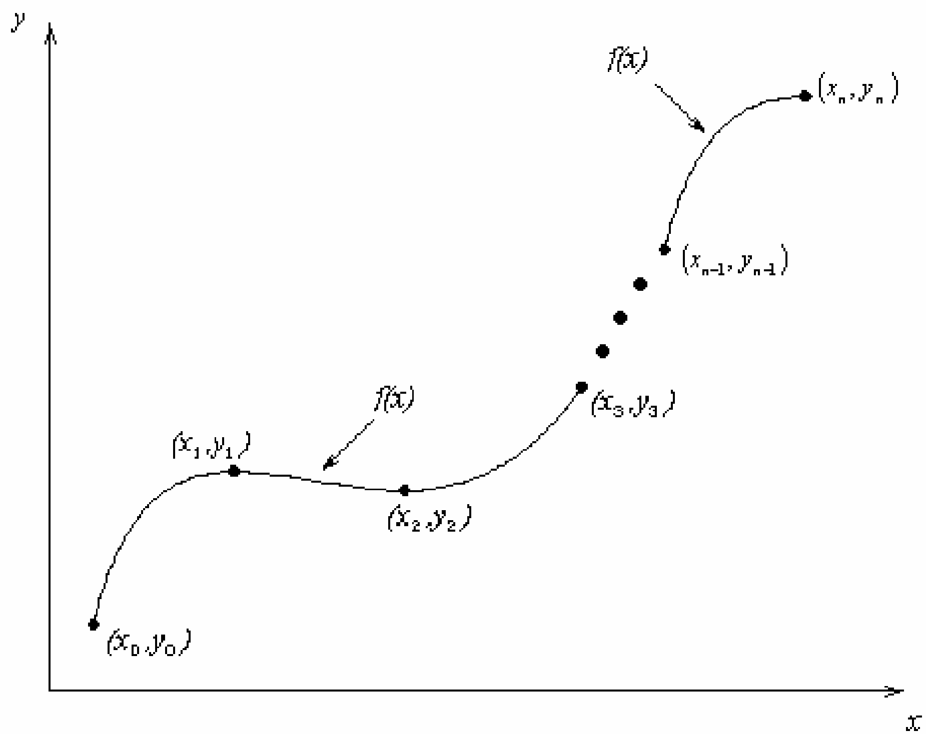
Interpolation

Topic: Direct Method

Major: Civil

What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Direct Method

Given 'n+1' data points (x_0,y_0), (x_1,y_1),............. (x_n,y_n), pass a polynomial of order 'n' through the data as given below:

$$y = a_0 + a_1 x + ...+ a_n x^n.$$

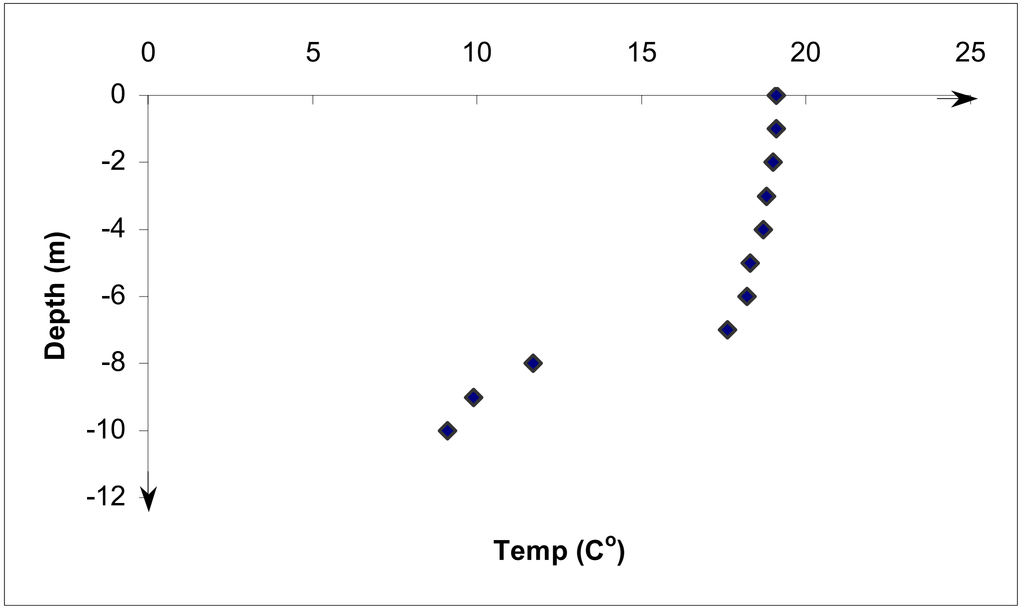
where a_0, a_1,.................. a_n are real constants.

- Set up 'n+1' equations to find 'n+1' constants.
- To find the value 'y' at a given value of 'x', simply substitute the value of 'x' in the above polynomial.

Example

We are given the temperature vs. depth plot for a lake. Determine the value of the temperature at $z = -7.5$ using the direct method for linear interpolation.

Temperature °C	Depth m
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10



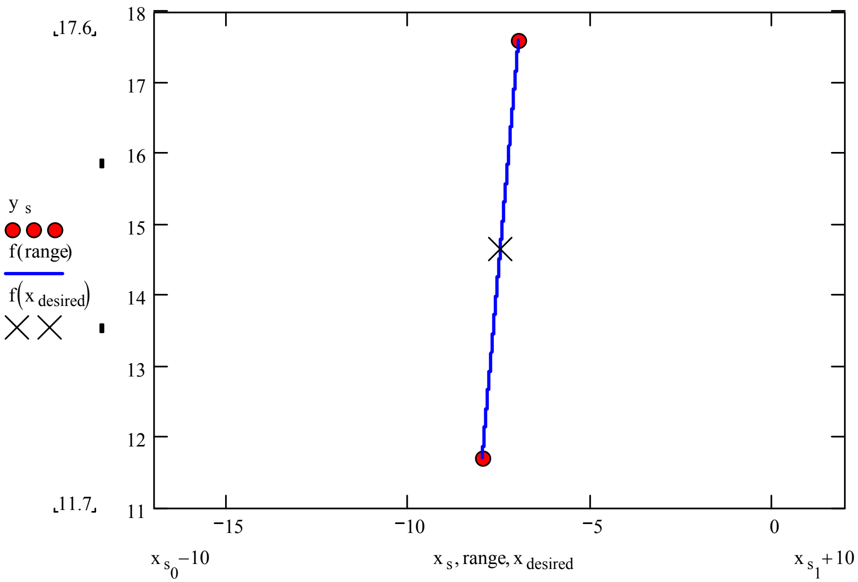
Temperature vs. depth of a lake

Linear Interpolation

$$T(z) = a_0 + a_1 z$$

$$T(-8) = a_0 + a_1(-8) = 11.7$$

$$T(-7) = a_0 + a_1(-7) = 17.6$$



Solving the above two equations gives,

$$a_0 = 58.9 \qquad a_1 = 5.9$$

Hence

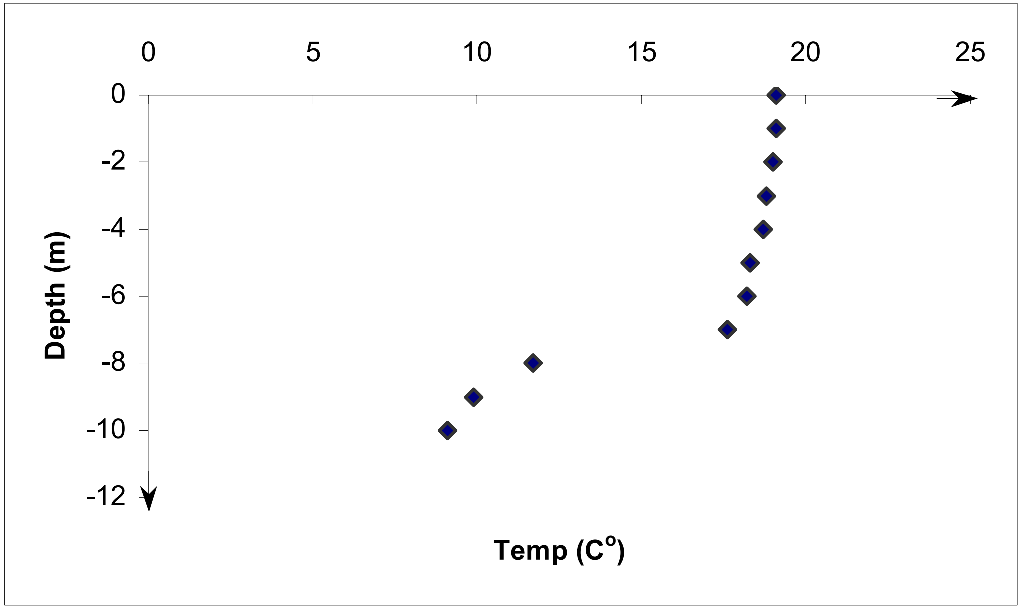
$$T(z) = 58.9 + 5.9z, \quad -7 \le z \le -8.$$

$$T(-7.5) = 58.9 + 5.9(-7.5) = 14.65^\circ C$$

Example

We are given the temperature vs. depth plot for a lake. Determine the value of the temperature at z = -7.5 using the direct method for quadratic interpolation.

Temperature	Depth
°C	m
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10



Temperature vs. depth of a lake

Quadratic Interpolation

$$T(z) = a_0 + a_1 z + a_2 z^2$$

$$T(-9) = a_0 + a_1(-9) + a_2(-9)^2 = 9.9$$

$$T(-8) = a_0 + a_1(-8) + a_2(-8)^2 = 11.7$$

$$T(-7) = a_0 + a_1(-7) + a_2(-7)^2 = 17.6$$

Solving the above three equations gives

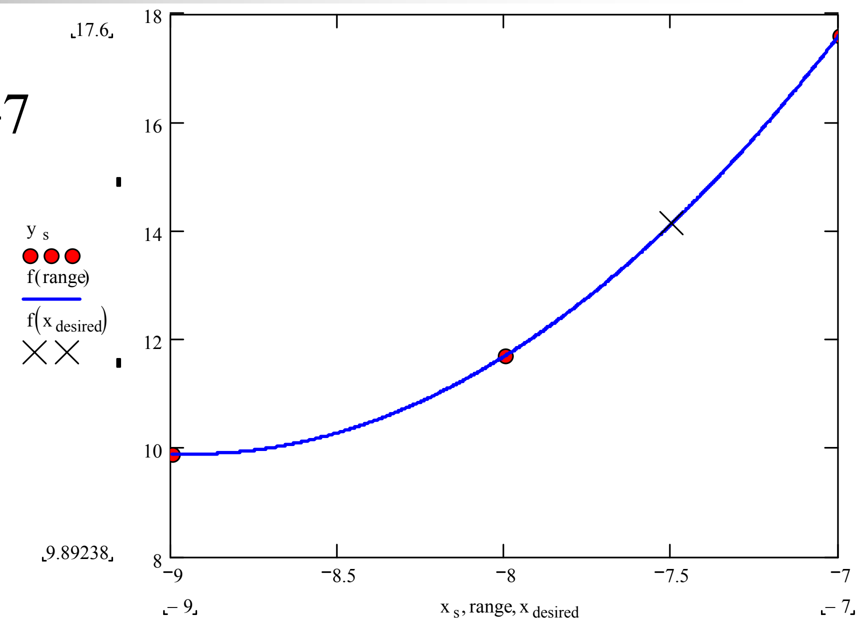
$$a_0 = 173.7 \qquad a_1 = 36.65 \qquad a_2 = 2.05$$

Quadratic Interpolation (contd)

$$T(z) = 173.7 + 36.65z + 2.05z^2, \quad -9 \le z \le -7$$

$$T(-7.5) = 173.7 + 36.65(-7.5) + 2.05(-7.5)^2$$

$$= 14.138°C$$



The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

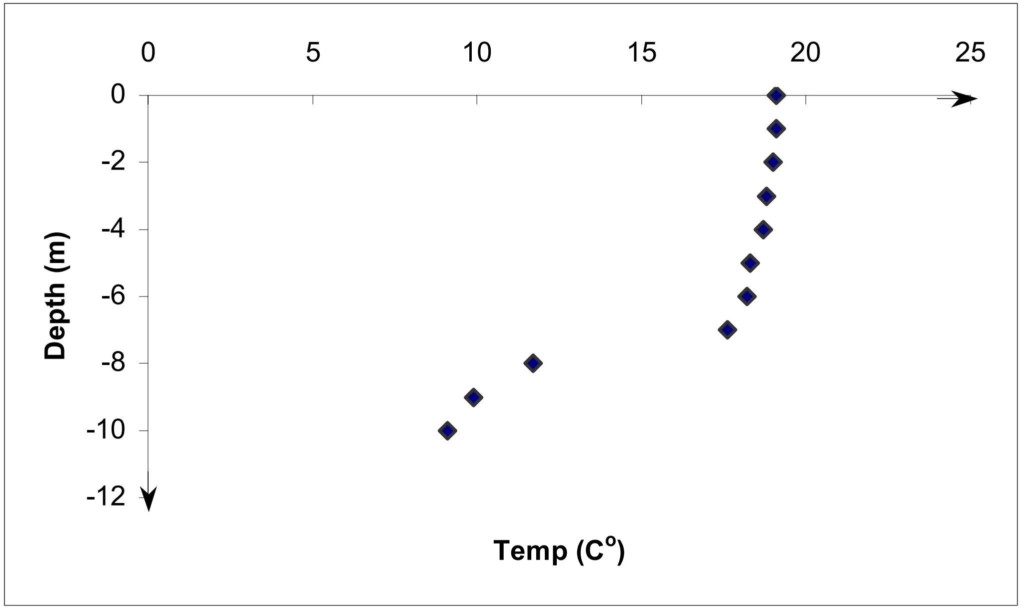
$$|\in_a| = \left|\frac{14.138 - 14.65}{14.138}\right| \times 100$$

$$= 3.6251\%$$

Example

We are given the temperature vs. depth plot for a lake. Determine the value of the temperature at $z = -7.5$ using the direct method for cubic interpolation.

Temperature	Depth
°C	m
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10



Temperature vs. depth of a lake

Cubic Interpolation

$$T(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$$

$$T(-9) = 9.9 = a_0 + a_1(-9) + a_2(-9)^2 + a_3(-9)^3$$

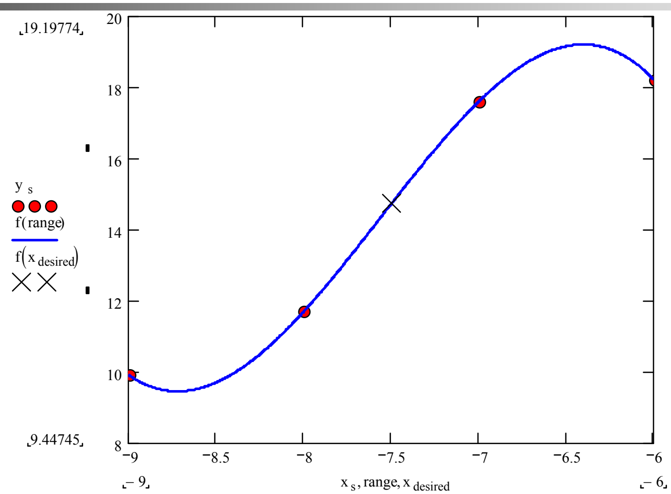
$$T(-8) = 11.7 = a_0 + a_1(-8) + a_2(-8)^2 + a_3(-8)^3$$

$$T(-7) = 17.6 = a_0 + a_1(-7) + a_2(-7)^2 + a_3(-7)^3$$

$$T(-6) = 18.2 = a_0 + a_1(-6) + a_2(-6)^2 + a_3(-6)^3$$

$$a_0 = -615.9 \quad a_1 = -262.58 \quad a_2 = -35.55 \quad a_3 = -1.5667$$

Cubic Interpolation (contd)



$$T(z) = -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \quad -9 \le z \le -6$$

$$T(-7.5) = -615.9 - 262.58(-7.5) - 35.55(-7.5)^2 - 1.5667(-7.5)^3 = 14.725°C$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a| = \left|\frac{14.725 - 14.138}{14.725}\right| \times 100$$

$$= 3.9898\%$$

Comparison Table

Order of Polynomial	1	2	3
Temperature °C	14.65	14.138	14.725
Absolute Relative Approximate Error	----------	3.6251%	3.9898%

Thermocline

What is the value of depth at which the thermocline exists?

The position where the thermocline exists is given where $\frac{d^2T}{dz^2} = 0$

$$T(z) = -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \quad -9 \le z \le -6$$

$$\frac{dT}{dz} = -262.58 - 71.10z - 4.7z^2, \quad -9 \le z \le -6$$

$$\frac{d^2T}{dz^2} = -71.10 - 9.4z, \quad -9 \le z \le -6$$

Simply setting this expression equal to zero, we get

$$0 = -71.10 - 9.4z, \quad -9 \le z \le -6$$

$$z = -7.5638\ m$$