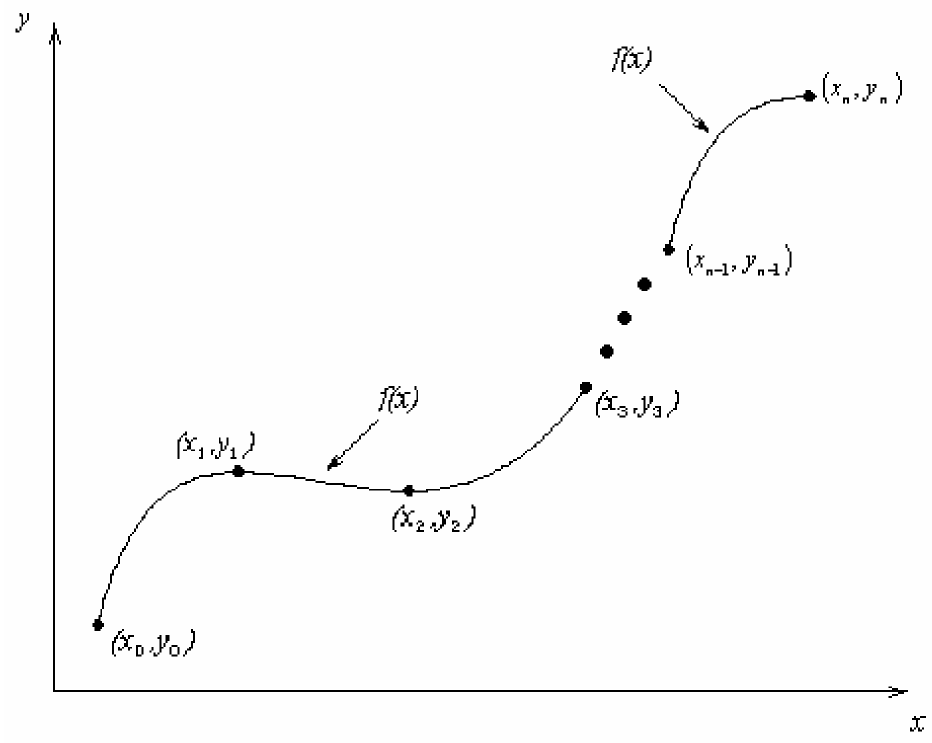
# Interpolation

## Topic: Lagrangian Interpolation

Major: Civil

# What is Interpolation ?

Given $(x_0,y_0)$, $(x_1,y_1)$, …… $(x_n,y_n)$, find the value of 'y' at a value of 'x' that is not given.



# Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

# Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where '$n$' in $f_n(x)$ stands for the $n^{th}$ order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

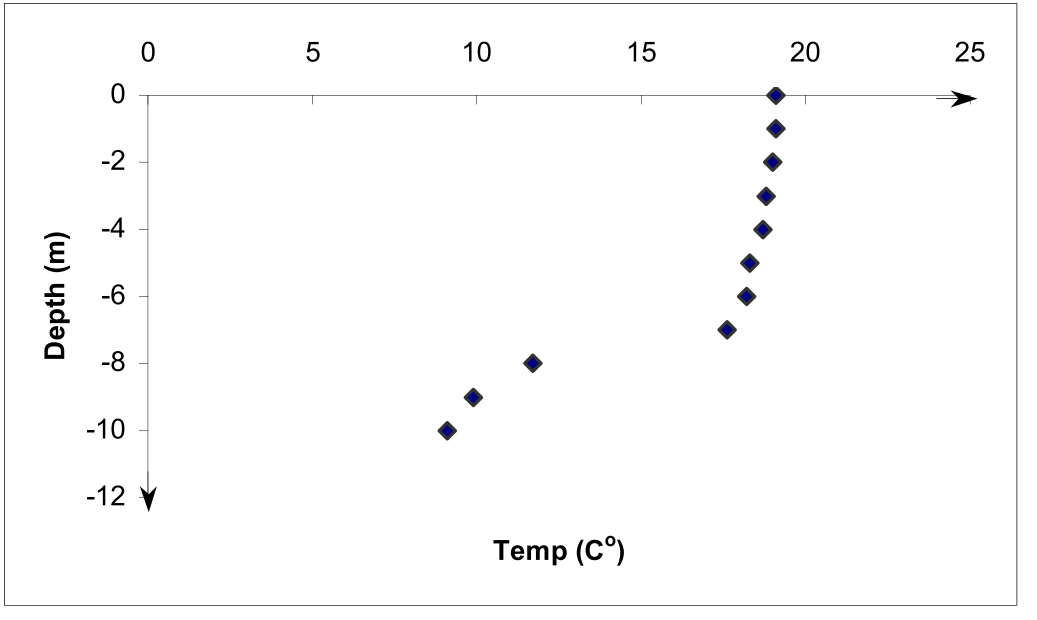
$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

# Example

We are given the temperature vs. depth plot for a lake. Determine the value of the temperature at $z = -7.5$ using the Lagrangian method for linear interpolation.

| Temperature $^oC$ | Depth m |
|---|---|
| 19.1 | 0 |
| 19.1 | -1 |
| 19 | -2 |
| 18.8 | -3 |
| 18.7 | -4 |
| 18.3 | -5 |
| 18.2 | -6 |
| 17.6 | -7 |
| 11.7 | -8 |
| 9.9 | -9 |
| 9.1 | -10 |




**Temperature vs. depth of a lake**

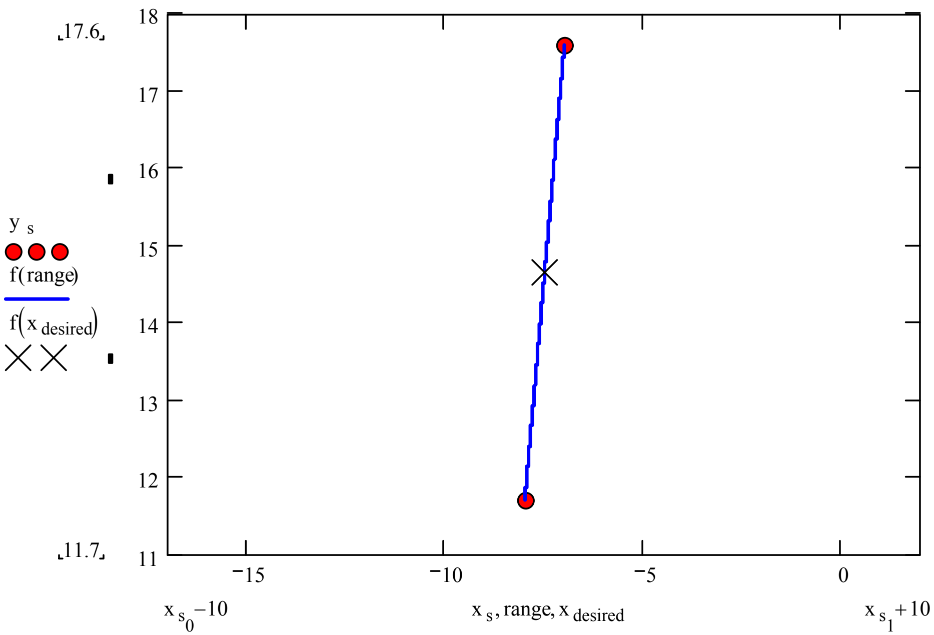
# Linear Interpolation

$$T(z) = \sum_{i=0}^{1} L_i(z) T(z_i)$$

$$= L_0(z)T(z_0) + L_1(z)T(z_1)$$

$$z_0 = -8, T(z_0) = 11.7$$

$$z_1 = -7, T(z_1) = 17.6$$



# Linear Interpolation (contd)

$$L_0(z) = \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{z - z_j}{z_0 - z_j} = \frac{z - z_1}{z_0 - z_1}$$

$$L_1(z) = \prod_{\substack{j=0 \\ j \neq 1}}^{1} \frac{z - z_j}{z_1 - z_j} = \frac{z - z_0}{z_1 - z_0}$$

$$T(z) = \frac{z - z_1}{z_0 - z_1} T(z_0) + \frac{z - z_0}{z_1 - z_0} T(z_1) = \frac{z + 7}{-8 + 7}(11.7) + \frac{z + 8}{-7 + 8}(17.6)$$

$$T(-7.5) = \frac{-7.5 + 7}{-8 + 7}(11.7) + \frac{-7.5 + 8}{-7 + 8}(17.6)$$

$$= 0.5(11.7) + 0.5(17.6)$$

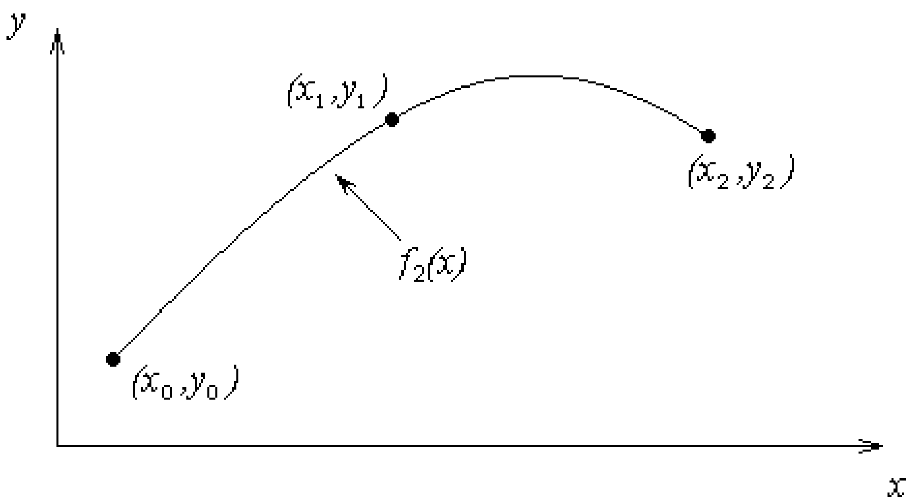
$$= 14.65°C$$

# Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the temperature given by

$$T(z) = \sum_{i=0}^{2} L_i(z) T(z_i)$$

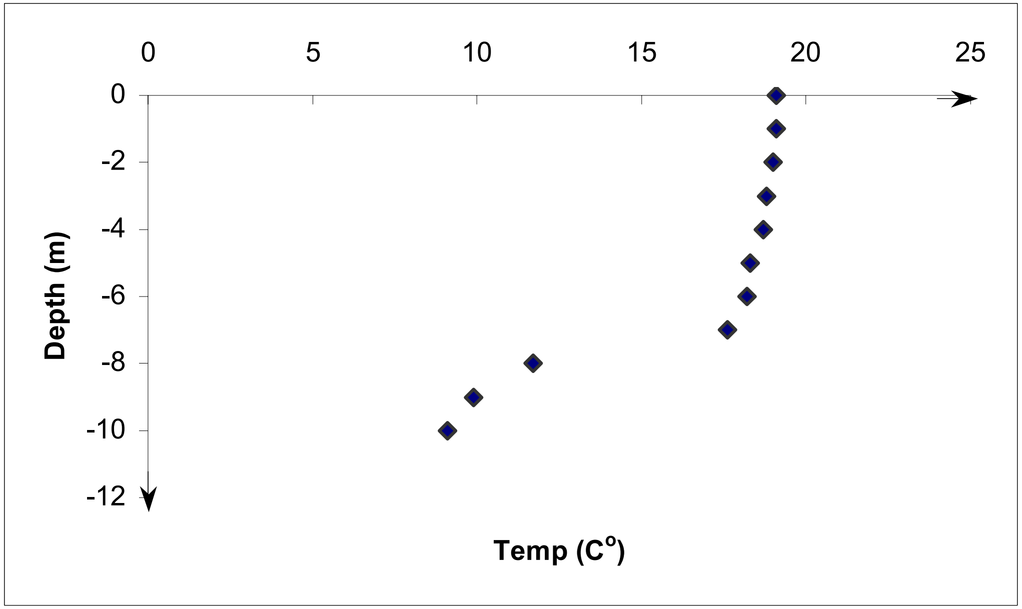
$$= L_0(z)T(z_0) + L_1(z)T(z_1) + L_2(z)T(z_2)$$



# Example

We are given the temperature vs. depth plot for a lake. Determine the value of the temperature at z = -7.5 using the Lagrangian method for quadratic interpolation.

| Temperature °C | Depth m |
|---|---|
| 19.1 | 0 |
| 19.1 | -1 |
| 19 | -2 |
| 18.8 | -3 |
| 18.7 | -4 |
| 18.3 | -5 |
| 18.2 | -6 |
| 17.6 | -7 |
| 11.7 | -8 |
| 9.9 | -9 |
| 9.1 | -10 |




**Temperature vs. depth of a lake**

# Quadratic Interpolation (contd)

$$z_o = -9,\ T(z_o) = 9.9$$

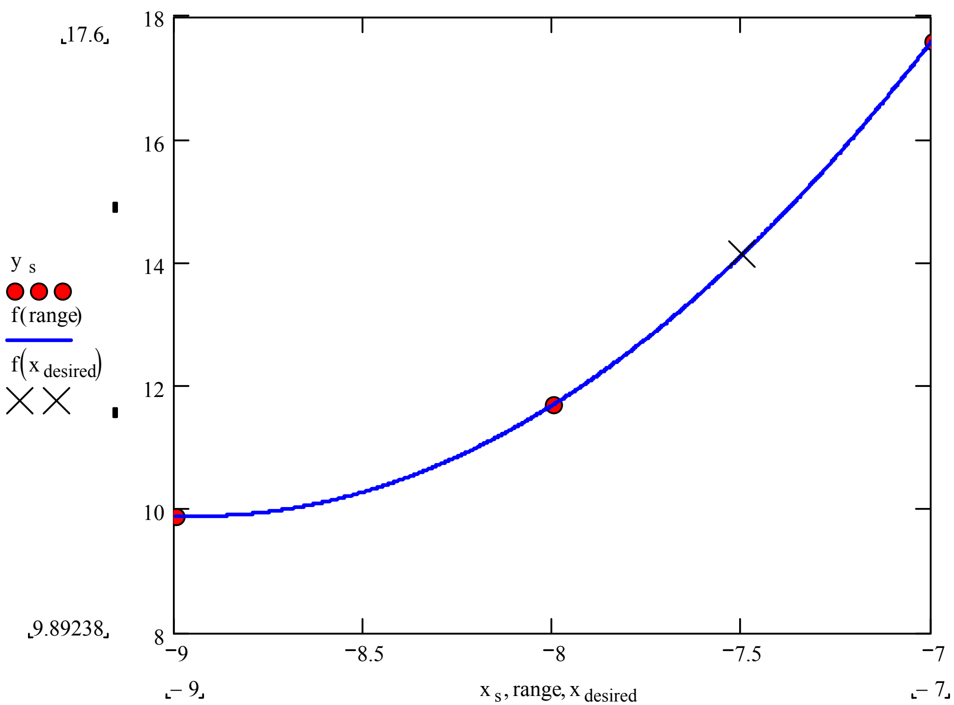
$$z_1 = -8,\ T(z_1) = 11.7$$

$$z_2 = -7,\ T(z_2) = 17.6$$

$$L_0(z) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{z - z_j}{z_0 - z_j} = \left(\frac{z - z_1}{z_0 - z_1}\right)\left(\frac{z - z_2}{z_0 - z_2}\right)$$

$$L_1(z) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{z - z_j}{z_1 - z_j} = \left(\frac{z - z_0}{z_1 - z_0}\right)\left(\frac{z - z_2}{z_1 - z_2}\right)$$

$$L_2(z) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{z - z_j}{z_2 - z_j} = \left(\frac{z - z_0}{z_2 - z_0}\right)\left(\frac{z - z_1}{z_2 - z_1}\right)$$



# Quadratic Interpolation (contd)

$$T(z)=\left(\frac{z-z_1}{z_0-z_1}\right)\left(\frac{z-z_2}{z_0-z_2}\right)T(z_0)+\left(\frac{z-z_0}{z_1-z_0}\right)\left(\frac{z-z_2}{z_1-z_2}\right)T(z_1)+\left(\frac{z-z_0}{z_2-z_0}\right)\left(\frac{z-z_1}{z_2-z_1}\right)T(z_2)$$

$$T(-7.5)=\frac{(-7.5+8)(-7.5+7)}{(-9+8)(-9+7)}(9.9)+\frac{(-7.5+9)(-7.5+7)}{(-8+9)(-8+7)}(11.7)$$
$$+\frac{(-7.5+9)(-7.5+8)}{(-7+9)(-7+8)}(17.6)$$

$$=(-0.125)(9.9)+(0.75)(11.7)+(0.375)(17.6)$$

$$=14.138°C$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a|=\left|\frac{14.138-14.65}{14.138}\right|\times 100$$

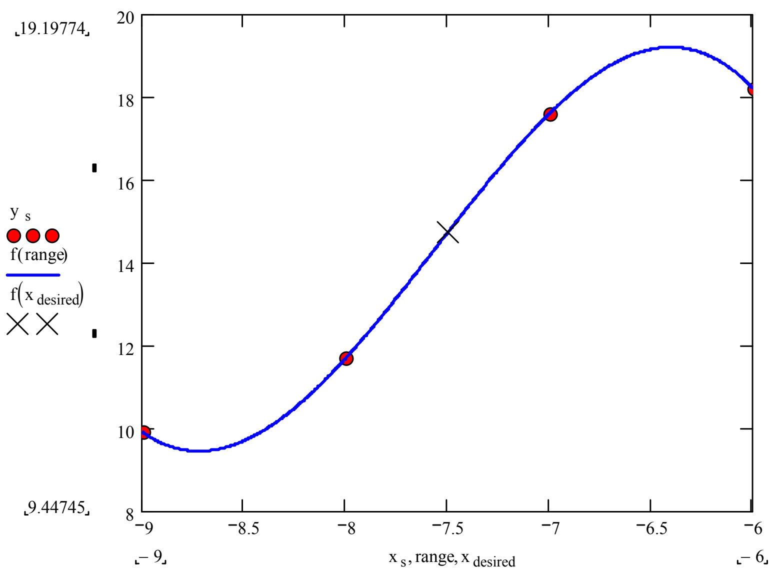
$$=3.6251\%$$

# Cubic Interpolation

For the third order polynomial (also called cubic interpolation), we choose the temperature given by

$$T(z) = \sum_{i=0}^{3} L_i(z)T(z_i)$$

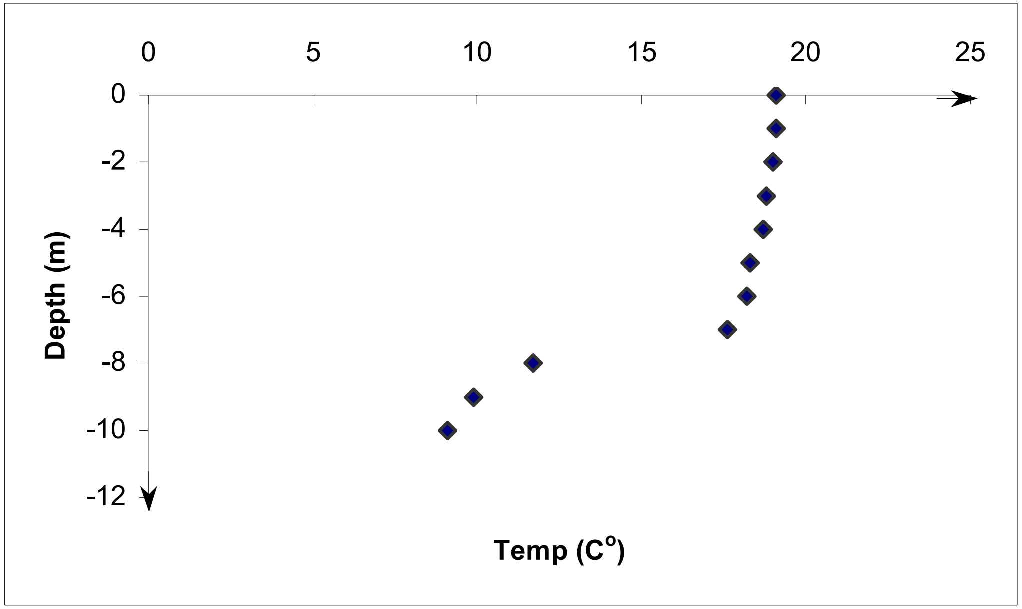
$$= L_0(z)T(z_0) + L_1(z)T(z_1) + L_2(z)T(z_2) + L_3(z)T(z_3)$$



# Example

We are given the temperature vs. depth plot for a lake. Determine the value of the temperature at z = -7.5 using the Lagrangian method for cubic interpolation.

| Temperature °C | Depth m |
|---|---|
| 19.1 | 0 |
| 19.1 | -1 |
| 19 | -2 |
| 18.8 | -3 |
| 18.7 | -4 |
| 18.3 | -5 |
| 18.2 | -6 |
| 17.6 | -7 |
| 11.7 | -8 |
| 9.9 | -9 |
| 9.1 | -10 |




**Temperature vs. depth of a lake**

# Cubic Interpolation (contd)

$z_o = -9, \; T(z_o) = 9.9$ $\qquad$ $z_1 = -8, \; T(z_1) = 11.7$

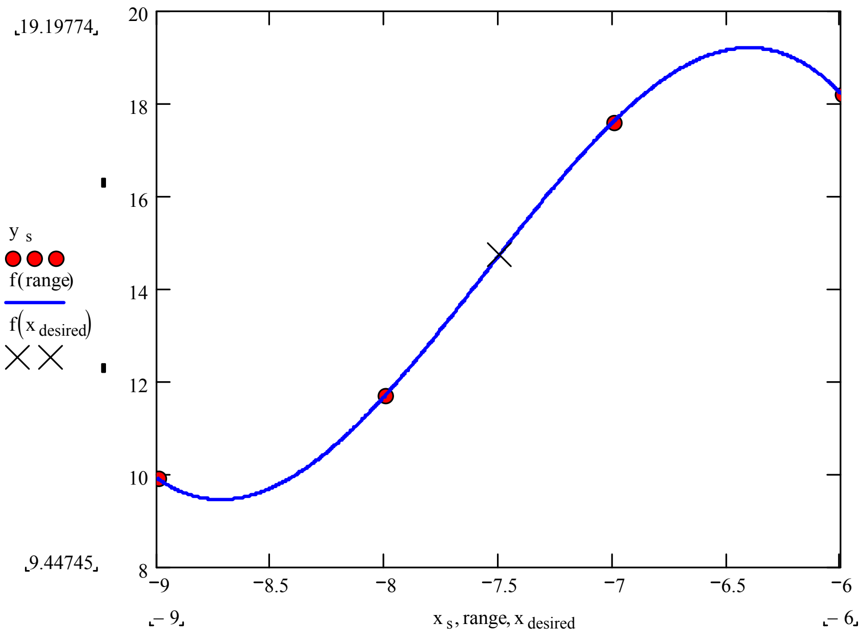
$z_2 = -7, \; T(z_2) = 17.6$ $\qquad$ $z_3 = -6, \; T(z_3) = 18.2$

$$L_0(z) = \prod_{\substack{j=0 \\ j \neq 0}}^{3} \frac{z - z_j}{z_0 - z_j} = \left(\frac{z - z_1}{z_0 - z_1}\right)\left(\frac{z - z_2}{z_0 - z_2}\right)\left(\frac{z - z_3}{z_0 - z_3}\right)$$

$$L_1(z) = \prod_{\substack{j=0 \\ j \neq 1}}^{3} \frac{z - z_j}{z_1 - z_j} = \left(\frac{z - z_0}{z_1 - z_0}\right)\left(\frac{z - z_2}{z_1 - z_2}\right)\left(\frac{z - z_3}{z_1 - z_3}\right)$$

$$L_2(z) = \prod_{\substack{j=0 \\ j \neq 2}}^{3} \frac{z - z_j}{z_2 - z_j} = \left(\frac{z - z_0}{z_2 - z_0}\right)\left(\frac{z - z_1}{z_2 - z_1}\right)\left(\frac{z - z_3}{z_2 - z_3}\right)$$

$$L_3(z) = \prod_{\substack{j=0 \\ j \neq 3}}^{3} \frac{z - z_j}{z_3 - z_j} = \left(\frac{z - z_0}{z_3 - z_0}\right)\left(\frac{z - z_1}{z_3 - z_1}\right)\left(\frac{z - z_2}{z_3 - z_2}\right)$$



# Cubic Interpolation (contd)

$$T(z)=\left(\frac{z-z_1}{z_0-z_1}\right)\left(\frac{z-z_2}{z_0-z_2}\right)\left(\frac{z-z_3}{z_0-z_3}\right)T(z_0)+\left(\frac{z-z_0}{z_1-z_0}\right)\left(\frac{z-z_2}{z_1-z_2}\right)\left(\frac{z-z_3}{z_1-z_3}\right)T(z_1)$$

$$+\left(\frac{z-z_0}{z_2-z_0}\right)\left(\frac{z-z_1}{z_2-z_1}\right)\left(\frac{z-z_3}{z_2-z_3}\right)T(z_2)+\left(\frac{z-z_0}{z_3-z_0}\right)\left(\frac{z-z_1}{z_3-z_1}\right)\left(\frac{z-z_2}{z_3-z_2}\right)T(z_3)$$

$$T(-7.5)=\frac{(-7.5+8)(-7.5+7)(-7.5+6)}{(-9+8)(-9+7)(-9+6)}(9.9)+\frac{(-7.5+9)(-7.5+7)(-7.5+6)}{(-8+9)(-8+7)(-7.5+6)}(11.7)$$

$$+\frac{(-7.5+9)(-7.5+8)(-7.5+6)}{(-7+9)(-7+8)(-7+6)}(17.6)+\frac{(-7.5+9)(-7.5+8)(-7.5+7)}{(-6+9)(-6+8)(-6+7)}(18.2)$$

$$=(-0.0625)(9.9)+(0.5625)(11.7)+(0.5625)(17.6)+(-0.0625)(18.2)$$

$$=14.725°C$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a|=\left|\frac{14.725-14.138}{14.725}\right|\times 100$$

$$=3.9898\%$$

# Comparison Table

| Order of Polynomial | 1 | 2 | 3 |
|---|---|---|---|
| Temperature °C | 14.65 | 14.138 | 14.725 |
| Absolute Relative Approximate Error | ---------- | 3.6251% | 3.9898% |

# Thermocline

What is the value of depth at which the thermocline exists?

The position where the thermocline exists is given where $\frac{d^2T}{dz^2} = 0$

$$T(z) = \frac{(z+8)(z+7)(z+6)}{(-9+8)(-9+7)(-9+6)}(9.9) + \frac{(z+9)(z+7)(z+6)}{(-8+9)(-8+7)(-7.5+6)}(11.7)$$
$$+ \frac{(z+9)(z+8)(z+6)}{(-7+9)(-7+8)(-7+6)}(17.6) + \frac{(z+9)(z+8)(z+7)}{(-6+9)(-6+8)(-6+7)}(18.2) \qquad -9 \le z \le -6$$

$$= -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \quad -9 \le z \le -6$$

$$\frac{dT}{dz} = -262.58 - 71.10z - 4.7z^2, \quad -9 \le z \le -6$$

$$\frac{d^2T}{dz^2} = -71.10 - 9.4z, \quad -9 \le z \le -6$$

Simply setting this expression equal to zero, we get

$$0 = -71.10 - 9.4z, \quad -9 \le z \le -6$$

$$z = -7.5638\ m$$