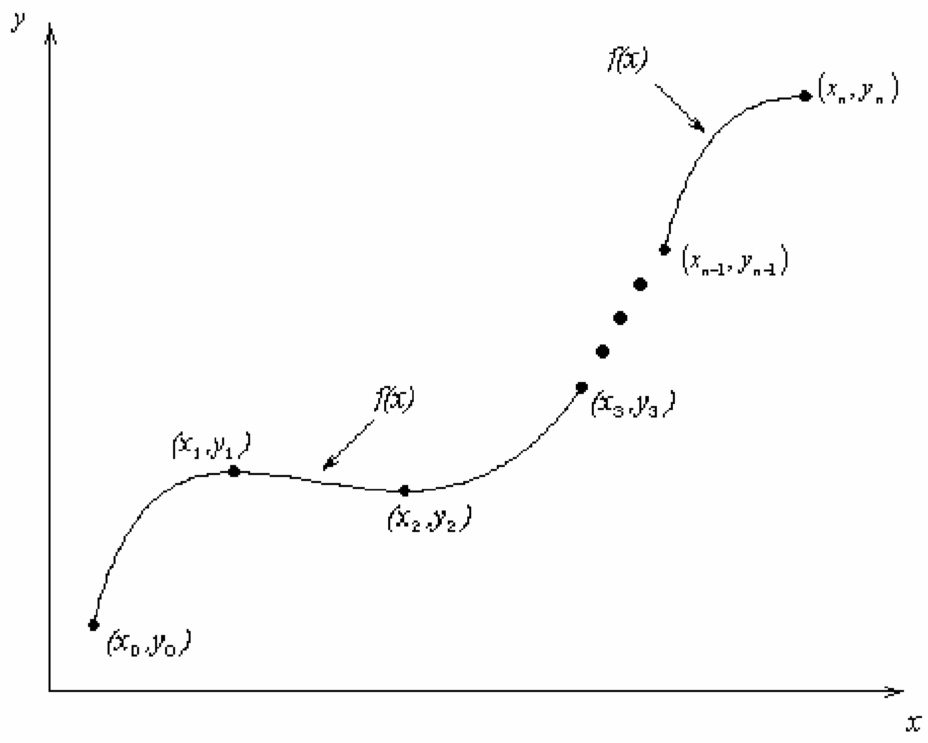
Interpolation

Topic: Newton's Divided Difference Polynomial Method

Major: Civil

What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), ... (x_n,y_n), find the value of ‘y’ at a value of ‘x’ that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Newton's Divided Difference Method

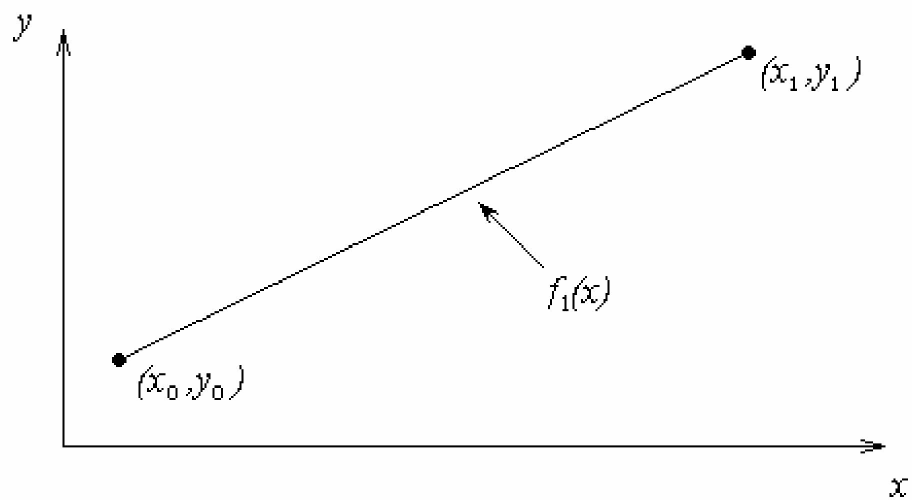
Linear interpolation: Given $(x_0, y_0), (x_1, y_1),$ pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$

where

$$b_0 = f(x_0)$$

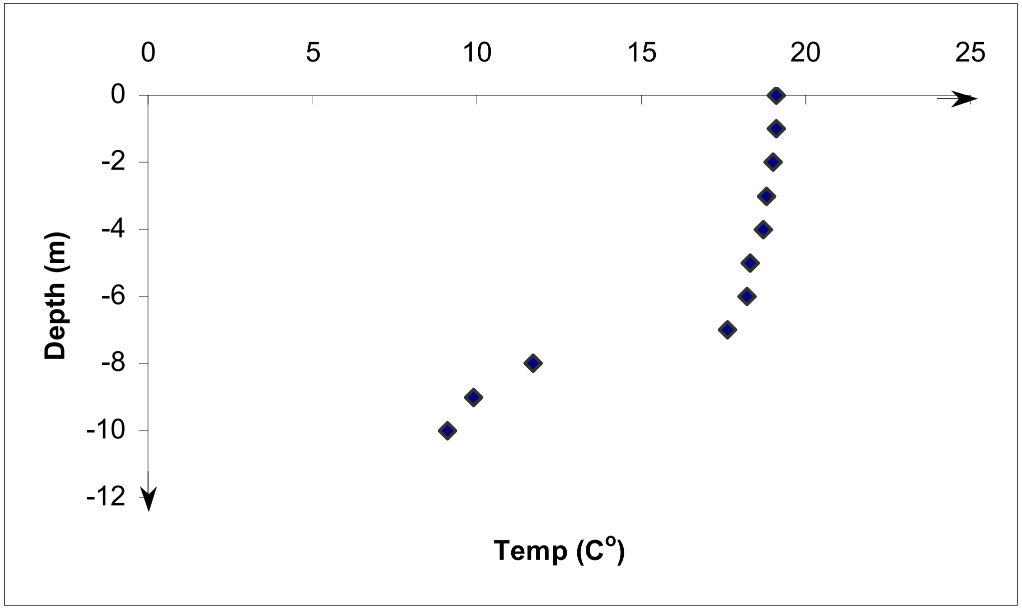
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



Example

Given the temperature vs. depth plot for a lake, determine the value of the temperature at depth z = -7.5 using the Newton Divided Difference method for linear interpolation.

Temperature °C	Depth m
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10



Temperature vs. depth of a lake

Linear Interpolation

$$T(z) = b_0 + b_1(z - z_0)$$

$$z_0 = -8, \; T(z_0) = 11.7$$

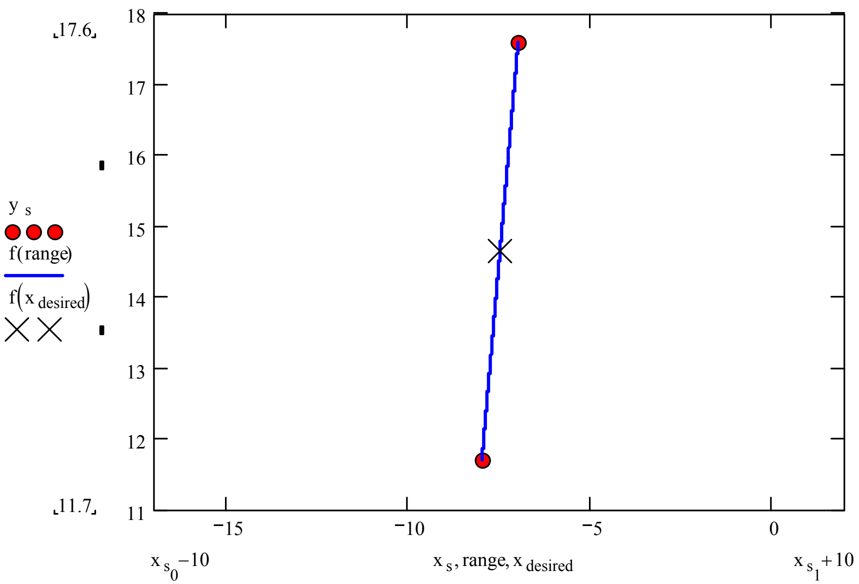
$$z_1 = -7, \; T(z_1) = 17.6$$

$$b_0 = T(z_0)$$

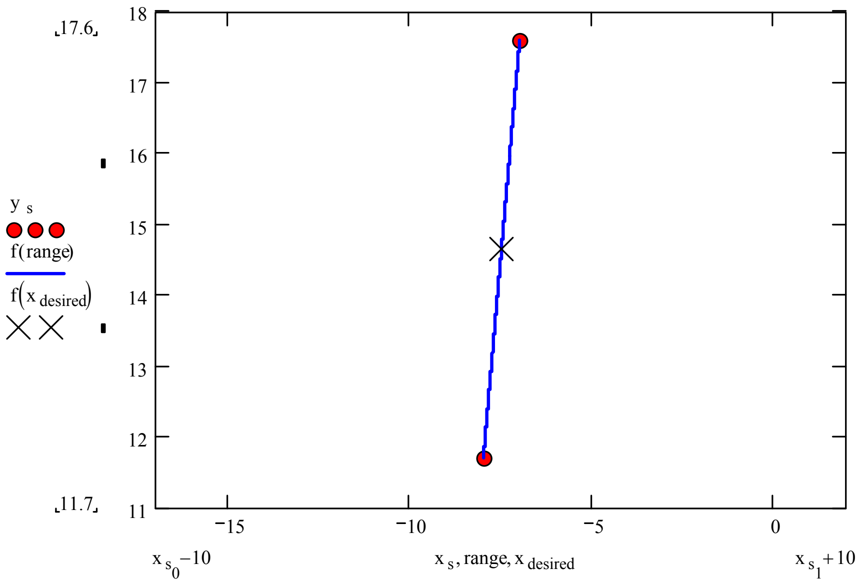
$$= 11.7$$

$$b_1 = \frac{T(z_1) - T(z_0)}{z_1 - z_0} = \frac{17.6 - 11.7}{-7 + 8}$$

$$= 5.9$$



Linear Interpolation (contd)



$$T(z) = b_0 + b_1(z - z_0)$$

$$= 11.7 + 5.9(z + 8), \qquad -8 \le z \le -7$$

At $z = -7.5$

$$T(-7.5) = 11.7 + 5.9(-7.5 + 8)$$

$$= 14.65°C$$

Quadratic Interpolation

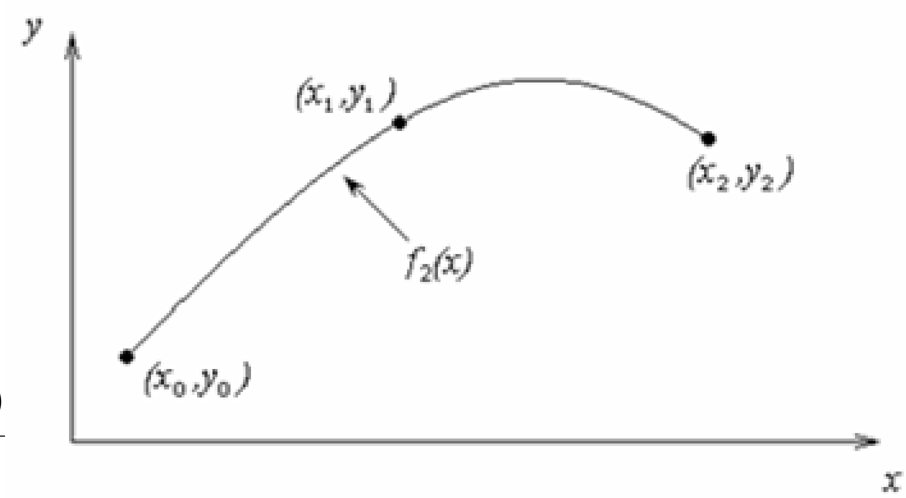
Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

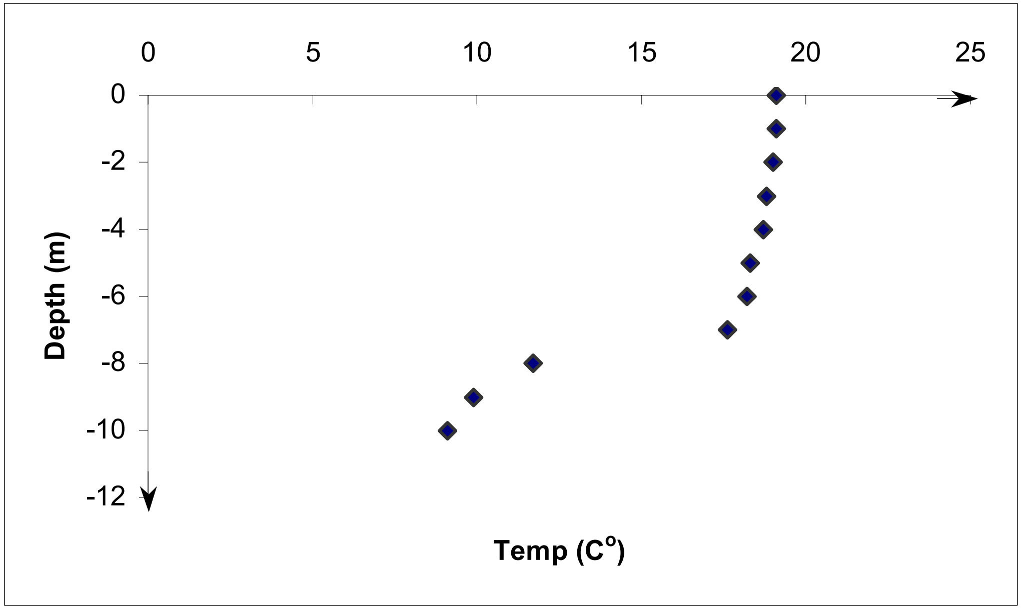
$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



Example

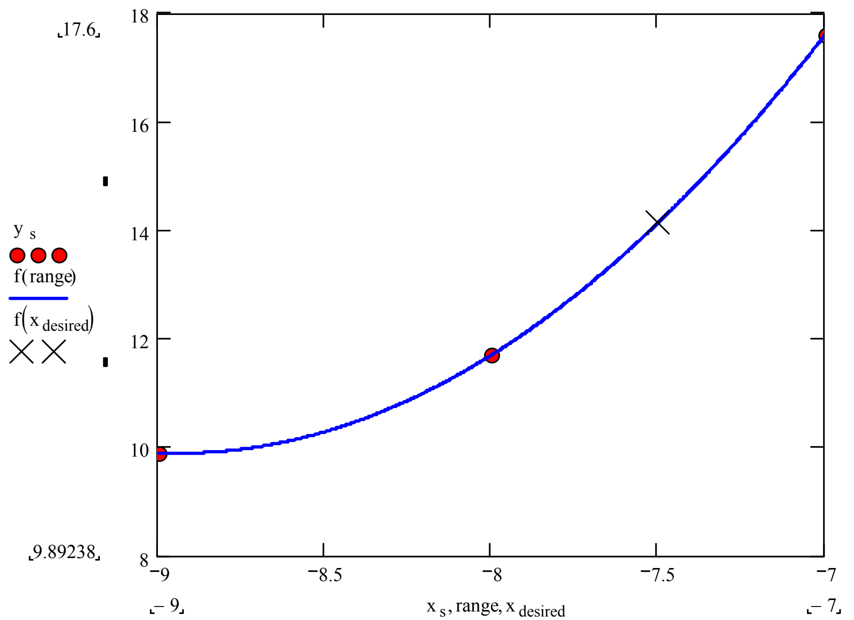
Given the temperature vs. depth plot for a lake, determine the value of the temperature at depth z = -7.5 using the Newton Divided Difference method for quadratic interpolation.

Temperature	Depth
°C	m
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10



Temperature vs. depth of a lake

Quadratic Interpolation (contd)



$$T(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1)$$

$$z_0 = -9,\ T(z_0) = 9.9$$

$$z_1 = -8,\ T(z_1) = 11.7$$

$$z_2 = -7,\ T(z_2) = 17.6$$

Quadratic Interpolation (contd)

$$b_0 = T(z_0)$$

$$= 9.9$$

$$b_1 = \frac{T(z_1) - T(z_0)}{z_1 - z_0} = \frac{11.7 - 9.9}{-8 + 7}$$

$$= 1.8$$

$$b_2 = \frac{\dfrac{T(z_2) - T(z_1)}{z_2 - z_1} - \dfrac{T(z_1) - T(z_0)}{z_1 - z_0}}{z_2 - z_0} = \frac{\dfrac{17.6 - 11.7}{-7 + 8} - \dfrac{11.7 - 9.9}{-8 + 9}}{-7 + 9}$$

$$= \frac{5.9 - 1.8}{2}$$

$$= 2.05$$

Quadratic Interpolation (contd)

$$T(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1)$$

$$= 9.9 + 1.8(z + 9) + 2.05(z + 9)(z + 8), \quad -9 \le z \le -7$$

At $z = -7.5,$

$$T(-7.5) = 9.9 + 1.8(-7.5 + 9) + 2.05(-7.5 + 9)(-7.5 + 8)$$

$$= 14.138°C$$

The absolute relative approximate error $|\in_a|$ obtained between the results

from the first and second order polynomial is

$$|\in_a| = \left|\frac{14.138 - 14.65}{14.138}\right| \times 100$$

$$= 3.6251\,\%$$

General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

General Form

Given $(n+1)$ data points, $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$,

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

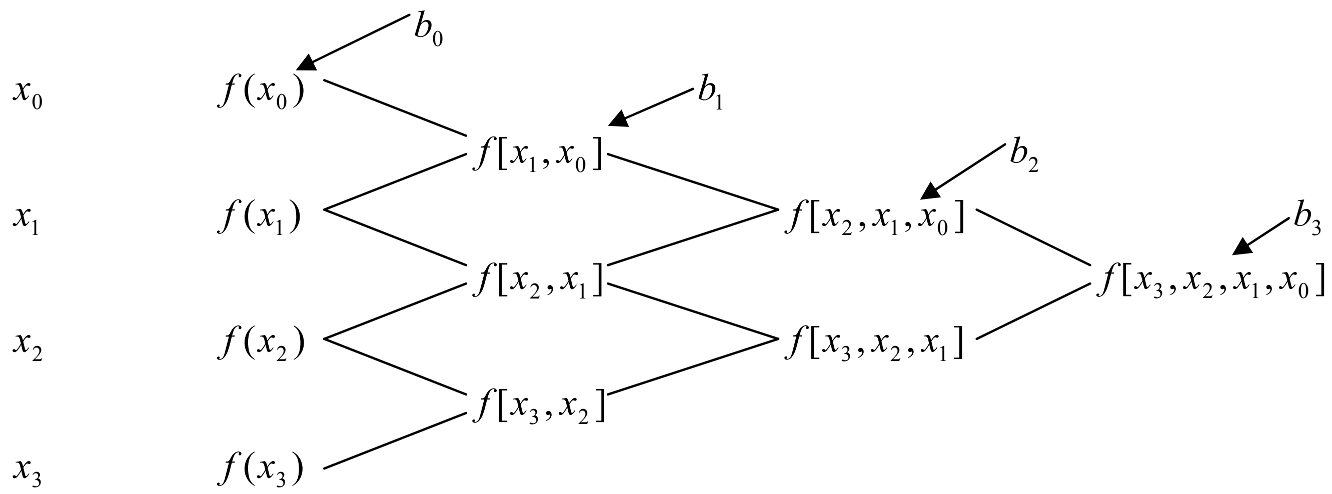
$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots., x_0]$$

$$b_n = f[x_n, x_{n-1}, \ldots., x_0]$$

General form

The third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3), is

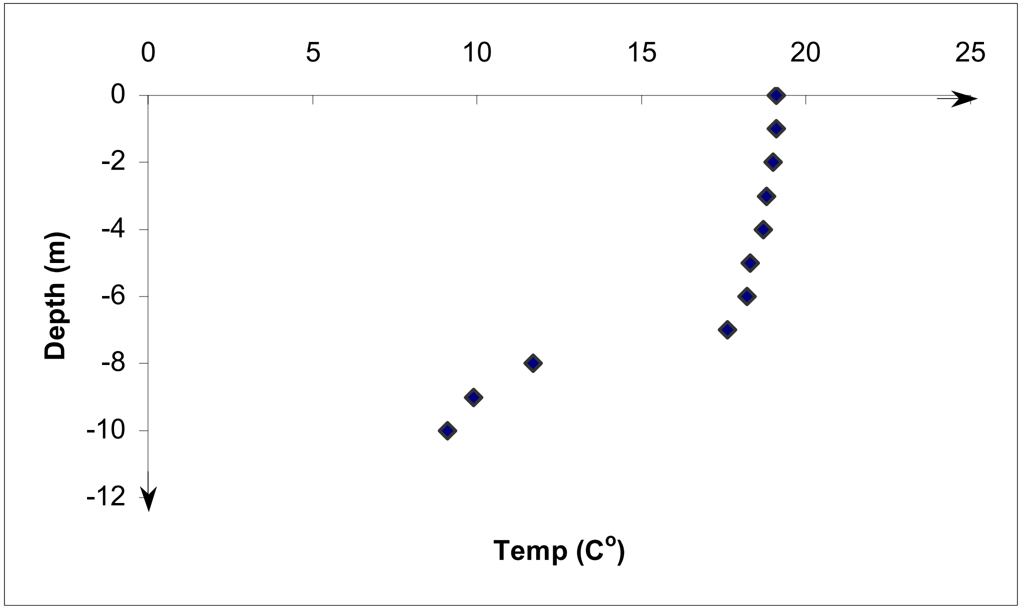
$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) \\ + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$



Example

Given the temperature vs. depth plot for a lake, determine the value of the temperature at depth z = -7.5 using the Newton Divided Difference method using cubic interpolation.

Temperature °C	Depth m
19.1	0
19.1	-1
19	-2
18.8	-3
18.7	-4
18.3	-5
18.2	-6
17.6	-7
11.7	-8
9.9	-9
9.1	-10



Temperature vs. depth of a lake

Example

The temperature profile is chosen as

$$T(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1) + b_3(z - z_0)(z - z_1)(z - z_2)$$

We need to choose four data points that are closest to $z = -7.5$

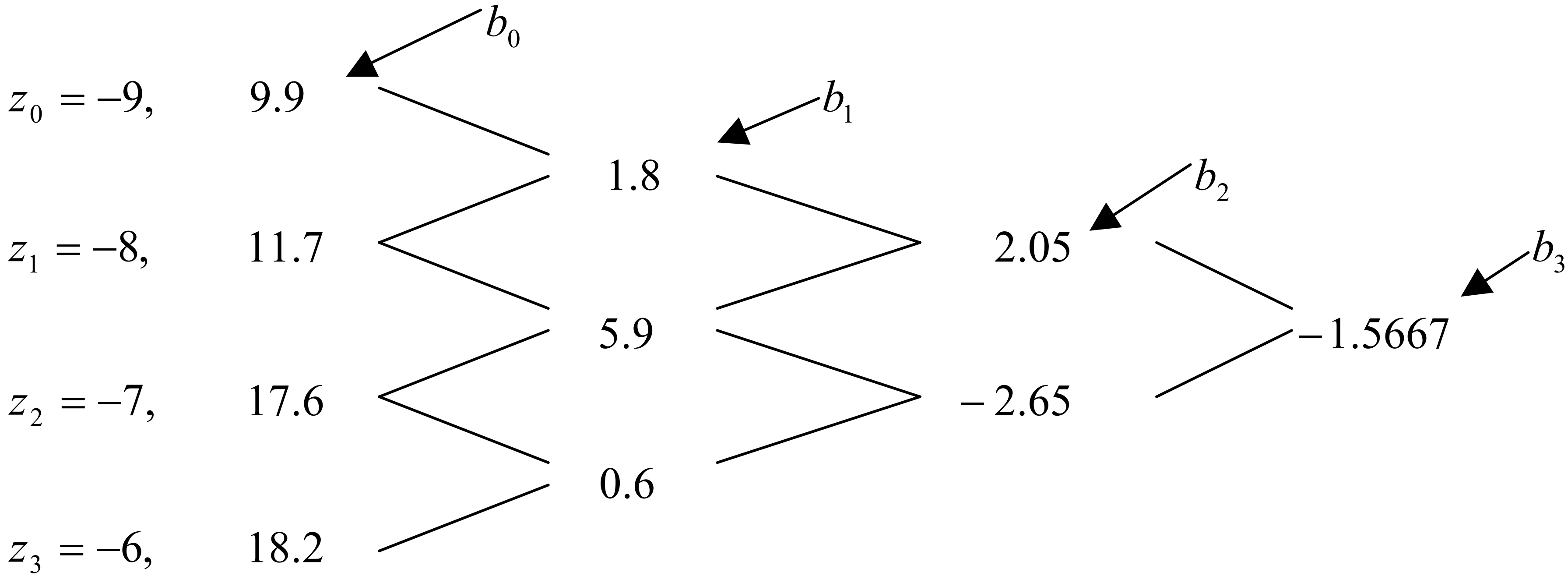
$$z_0 = -9, \quad T(z_0) = 9.9$$

$$z_1 = -8, \quad T(z_1) = 11.7$$

$$z_2 = -7, \quad T(z_2) = 17.6$$

$$z_3 = -6, \quad T(z_3) = 18.2$$

Example



The values of the constants are obtained as

$$b_0 = 9.9 \qquad b_1 = 1.8 \qquad b_2 = 2.05 \qquad b_3 = -1.5667$$

Example

$$T(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1) + b_3(z - z_0)(z - z_1)(z - z_2)$$

$$= 9.9 + 1.8(z + 9) + 2.05(z + 9)(z + 8) - 1.5667(z + 9)(z + 8)(z + 7)$$

At $z = -7.5$,

$$T(-7.5) = 9.9 + 1.8(-7.5 + 9) + 2.05(-7.5 + 9)(-7.5 + 8)$$
$$-1.5667(-7.5 + 9)(-7.5 + 8)(-7.5 + 7)$$

$$= 14.725°C$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a| = \left|\frac{14.725 - 14.138}{14.725}\right| \times 100$$

$$= 3.9898\%$$

Comparison Table

Order of Polynomial	1	2	3
Temperature °C	14.65	14.138	14.725
Absolute Relative Approximate Error	----------	3.6251%	3.9898%

Thermocline

What is the value of depth at which the thermocline exists?

The position where the thermocline exists is given where $\dfrac{d^2T}{dz^2} = 0$

$$T(z) = -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \quad -9 \le z \le -6$$

$$\frac{dT}{dz} = -262.58 - 71.10z - 4.7z^2, \quad -9 \le z \le -6$$

$$\frac{d^2T}{dz^2} = -71.10 - 9.4z, \quad -9 \le z \le -6$$

Simply setting this expression equal to zero, we get

$$0 = -71.10 - 9.4z, \quad -9 \le z \le -6$$

$$z = -7.5638 \ m$$