Chapter 05.03
Newton's Divided Difference Interpolation

After reading this chapter, you should be able to:

1. *derive Newton's divided difference method of interpolation,*
2. *apply Newton's divided difference method of interpolation, and*
3. *apply Newton's divided difference method interpolants to find derivatives and integrals.*

What is interpolation?

Many a times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x? Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called interpolation.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called extrapolation.

So what kind of function $f(x)$ should one choose? A polynomial is a common choice for an interpolating function because polynomials are easy to

(A) evaluate,
(B) differentiate, and
(C) integrate,

relative to other choices such as a trigonometric and exponential series.

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ points. One of the methods of interpolation is called Newton's divided difference polynomial method. Other methods include the direct method and the Lagrangian interpolation method. We discuss Newton's divided difference polynomial method in this chapter.

Newton's Divided Difference Polynomial Method

To illustrate this method, linear and quadratic interpolation is presented first. Then, the general form of Newton's divided difference polynomial method is presented. To illustrate the general form, cubic interpolation is shown in Figure 1.

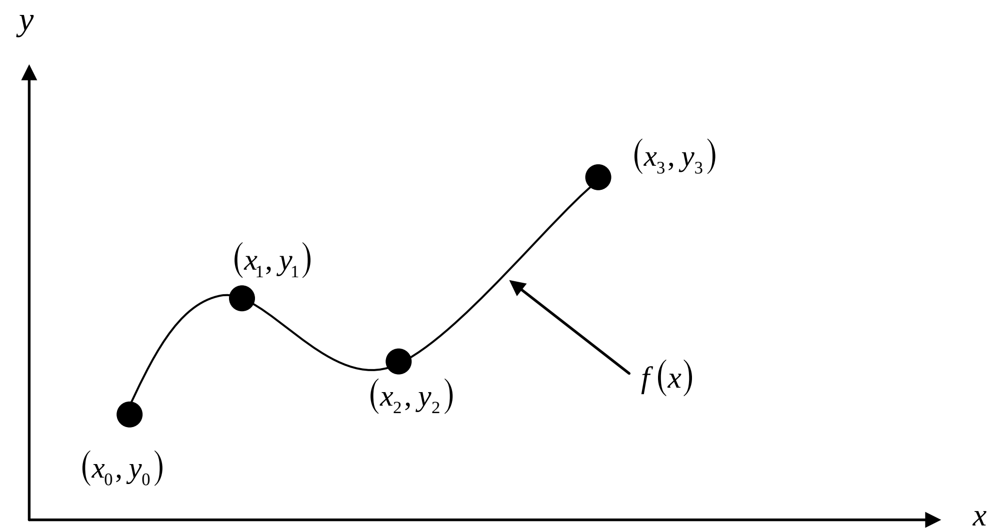

Figure 1 Interpolation of discrete data.

Linear Interpolation

Given (x_0, y_0) and (x_1, y_1), fit a linear interpolant through the data. Noting $y = f(x)$ and $y_1 = f(x_1)$, assume the linear interpolant $f_1(x)$ is given by (Figure 2)

$$f_1(x) = b_0 + b_1(x - x_0)$$

Since at $x = x_0$,

$$f_1(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) = b_0$$

and at $x = x_1$,

$$f_1(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0)$$
$$= f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So

$$b_0 = f(x_0)$$
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

giving the linear interpolant as

$$f_1(x) = b_0 + b_1(x - x_0)$$
$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

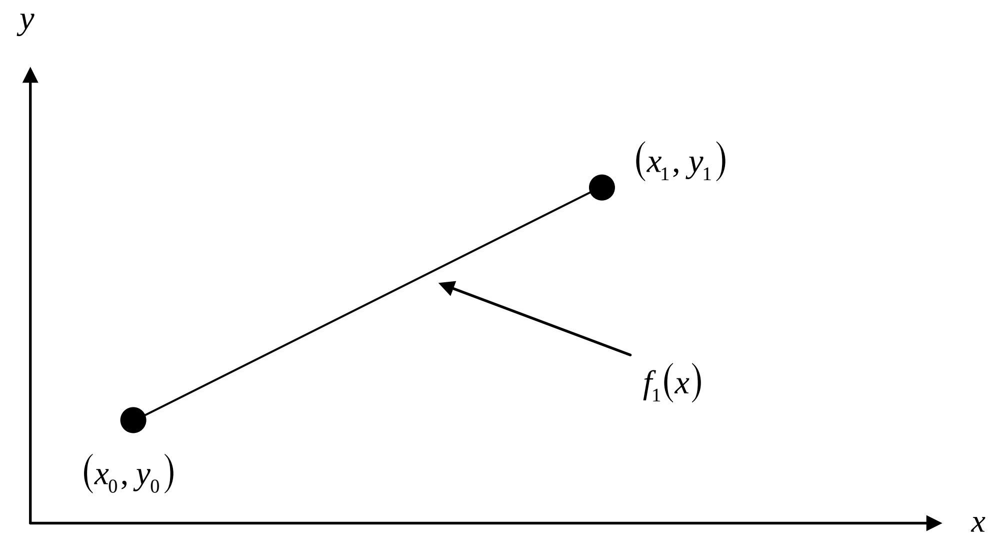

Figure 2 Linear interpolation.

Example 1

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth data for a lake in Table 1.

Table 1 Temperature vs. depth for a lake.

Temperature, T (°C)	Depth, z (m)
19.1	0
19.1	–1
19	–2
18.8	–3
18.7	–4
18.3	–5
18.2	–6
17.6	–7
11.7	–8
9.9	–9
9.1	–10

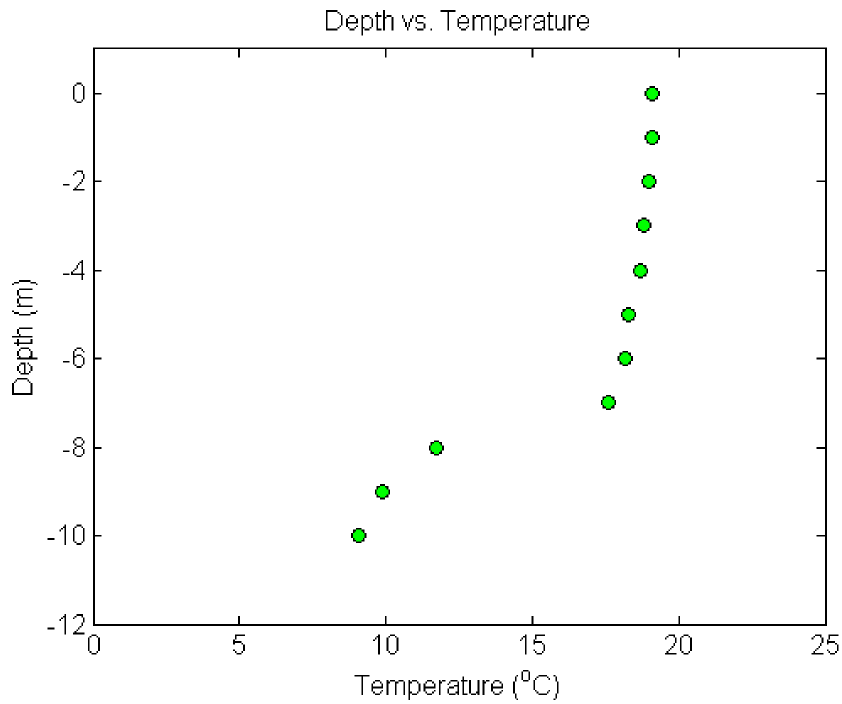

Figure 3 Temperature vs. depth of a lake.

Using the given data, we see the largest change in temperature is between $z=-8\text{ m}$ and $z=-7\text{ m}$. Determine the value of the temperature at $z=-7.5\text{ m}$ using Newton's divided difference method of interpolation and a first order polynomial.

Solution

For linear interpolation, the temperature is given by

$$T(z)=b_0+b_1(z-z_0)$$

Since we want to find the temperature at $z=-7.5\text{ m}$, and we are using a first order polynomial, we need to choose the two data points that are closest to $z=-7.5\text{ m}$ that also bracket $z=-7.5\text{ m}$ to evaluate it. The two points are $z_0=-8$ and $z_1=-7$.

Then

$$z_0=-8,\ T(z_0)=11.7$$

$$z_1=-7,\ T(z_1)=17.6$$

gives

$$b_0=T(z_0)$$

$$=11.7$$

$$b_1=\frac{T(z_1)-T(z_0)}{z_1-z_0}$$

$$=\frac{17.6-11.7}{-7+8}$$

$$= 5.9$$

Hence

$$T(z) = b_0 + b_1(z - z_0)$$
$$= 11.7 + 5.9(z + 8), \qquad -8 \le z \le -7$$

At $z = -7.5$

$$T(-7.5) = 11.7 + 5.9(-7.5 + 8)$$
$$= 14.65\,°\text{C}$$

If we expand

$$T(z) = 11.7 + 5.9(z + 8), \qquad -8 \le z \le -7$$

we get

$$T(z) = 58.9 + 5.9z, \qquad -8 \le z \le -7$$

This is the same expression as obtained in the direct method.

Quadratic Interpolation

Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data. Noting $y = f(x)$, $y_0 = f(x_0)$, $y_1 = f(x_1)$, and $y_2 = f(x_2)$, assume the quadratic interpolant $f_2(x)$ is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

At $x = x_0$,

$$f_2(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1)$$
$$= b_0$$
$$b_0 = f(x_0)$$

At $x = x_1$

$$f_2(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1)$$
$$f(x_1) = f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

At $x = x_2$

$$f_2(x_2) = f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$
$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

Giving

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Hence the quadratic interpolant is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}(x - x_0)(x - x_1)$$

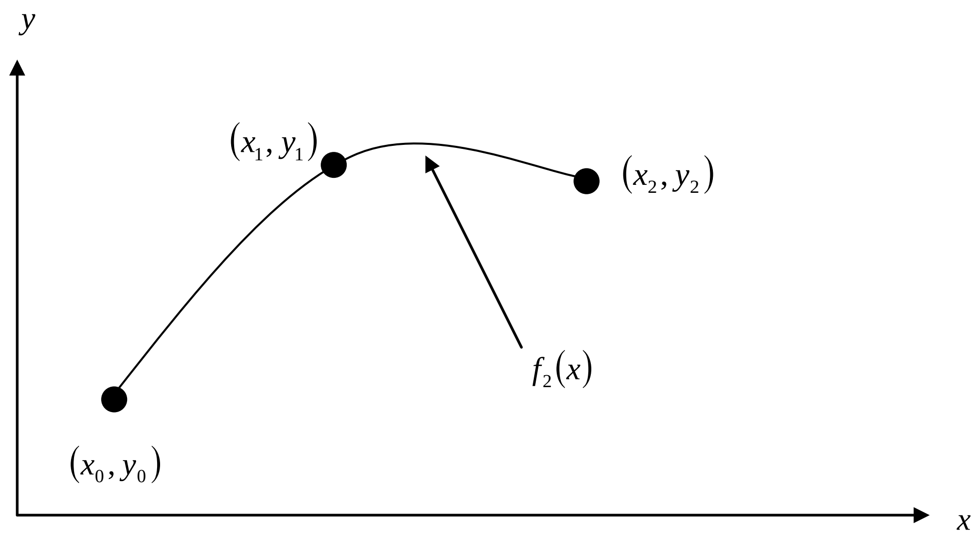

Figure 4 Quadratic interpolation.

Example 2

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth data for a lake in Table 2.

Table 2 Temperature vs. depth for a lake.

Temperature, T (°C)	Depth, z (m)
19.1	0
19.1	–1
19	–2
18.8	–3
18.7	–4
18.3	–5
18.2	–6
17.6	–7
11.7	–8
9.9	–9
9.1	–10

Using the given data, we see the largest change in temperature is between $z = -8$ m and $z = -7$ m. Determine the value of the temperature at $z = -7.5$ m using Newton's divided difference method of interpolation and a second order polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For quadratic interpolation, the temperature is given by

$$T(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1)$$

Since we want to find the temperature at $z = -7.5$, and we are using a second order polynomial, we need to choose the three data points that are closest to $z = -7.5$ that also bracket $z = -7.5$ to evaluate it. The three points are $z_0 = -9$, $z_1 = -8$ and $z_2 = -7$. (Choosing the three points as $z_0 = -8$, $z_1 = -7$ and $z_2 = -6$ is equally valid.)

Then

$$z_0 = -9,\ T(z_0) = 9.9$$
$$z_1 = -8,\ T(z_1) = 11.7$$
$$z_2 = -7,\ T(z_2) = 17.6$$

gives

$$\begin{aligned} b_0 &= T(z_0) \\ &= 9.9 \end{aligned}$$

$$\begin{aligned} b_1 &= \frac{T(z_1) - T(z_0)}{z_1 - z_0} \\ &= \frac{11.7 - 9.9}{-8 + 7} \\ &= 1.8 \end{aligned}$$

$$\begin{aligned} b_2 &= \frac{\dfrac{T(z_2) - T(z_1)}{z_2 - z_1} - \dfrac{T(z_1) - T(z_0)}{z_1 - z_0}}{z_2 - z_0} \\ &= \frac{\dfrac{17.6 - 11.7}{-7 + 8} - \dfrac{11.7 - 9.9}{-8 + 9}}{-7 + 9} \\ &= \frac{5.9 - 1.8}{2} \\ &= 2.05 \end{aligned}$$

Hence

$$\begin{aligned} T(z) &= b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1) \\ &= 9.9 + 1.8(z + 9) + 2.05(z + 9)(z + 8), \quad -9 \le z \le -7 \end{aligned}$$

At $z = -7.5$,

$$\begin{aligned} T(-7.5) &= 9.9 + 1.8(-7.5 + 9) + 2.05(-7.5 + 9)(-7.5 + 8) \\ &= 14.138\,^\circ\text{C} \end{aligned}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{14.138 - 14.65}{14.138}\right| \times 100$$
$$= 3.6251\%$$

If we expand

$$T(z) = 9.9 + 1.8(z+9) + 2.05(z+9)(z+8), \quad -9 \le z \le -7$$

we get

$$T(z) = 173.7 + 36.65z + 2.05z^2, \quad -9 \le z \le -7$$

This is the same expression obtained by the direct method.

General Form of Newton's Divided Difference Polynomial

In the two previous cases, we found linear and quadratic interpolants for Newton's divided difference method. Let us revisit the quadratic polynomial interpolant formula

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Note that b_0, b_1, and b_2 are finite divided differences. b_0, b_1, and b_2 are the first, second, and third finite divided differences, respectively. We denote the first divided difference by

$$f[x_0] = f(x_0)$$

the second divided difference by

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and the third divided difference by

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

where $f[x_0]$, $f[x_1, x_0]$, and $f[x_2, x_1, x_0]$ are called bracketed functions of their variables enclosed in square brackets.

Rewriting,

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

This leads us to writing the general form of the Newton's divided difference polynomial for $n+1$ data points, $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, as

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$
$$b_1 = f[x_1, x_0]$$
$$b_2 = f[x_2, x_1, x_0]$$
$$\vdots$$
$$b_{n-1} = f[x_{n-1}, x_{n-2}, ..., x_0]$$
$$b_n = f[x_n, x_{n-1}, ..., x_0]$$

where the definition of the m^{th} divided difference is

$$b_m = f[x_m, ..., x_0]$$
$$= \frac{f[x_m, ..., x_1] - f[x_{m-1}, ..., x_0]}{x_m - x_0}$$

From the above definition, it can be seen that the divided differences are calculated recursively.

For an example of a third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3),

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$
$$+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

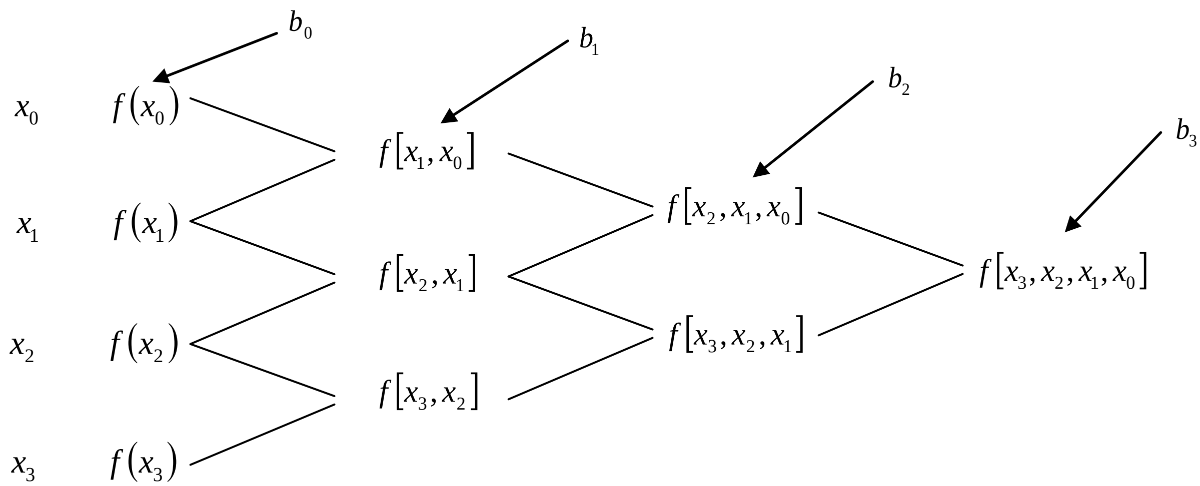

Figure 5 Table of divided differences for a cubic polynomial.

Example 3

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth data for a lake in Table 3.

Table 3 Temperature vs. depth for a lake.

Temperature, T $(°\text{C})$	Depth, z (m)
19.1	0
19.1	–1
19	–2
18.8	–3
18.7	–4
18.3	–5
18.2	–6
17.6	–7
11.7	–8
9.9	–9
9.1	–10

Using the given data, we see the largest change in temperature is between $z = -8\text{ m}$ and $z = -7\text{ m}$.

a) Determine the value of the temperature at $z = -7.5\text{ m}$ using the direct method of interpolation and a third order polynomial. Find the absolute relative approximate error for the third order polynomial approximation.

b) The position where the thermocline exists is given where $\frac{d^2T}{dz^2} = 0$. Using the expression from part (a), what is the value of the depth at which the thermocline exists?

Solution

a) For a third order polynomial, the temperature is given by

$$T(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1) + b_3(z - z_0)(z - z_1)(z - z_2)$$

Since we want to find the temperature at $z = -7.5$, and we are using a third order polynomial, we need to choose the four data points that are closest to $z = -7.5$ that also bracket $z = -7.5$ to evaluate it. The four points are $z_0 = -9$, $z_1 = -8$, $z_2 = -7$ and $z_3 = -6$.

Then

$$z_0 = -9, \quad T(z_0) = 9.9$$
$$z_1 = -8, \quad T(z_1) = 11.7$$
$$z_2 = -7, \quad T(z_2) = 17.6$$
$$z_3 = -6, \quad T(z_3) = 18.2$$

gives

$$\begin{aligned} b_0 &= T[z_0] \\ &= T(z_0) \\ &= 9.9 \end{aligned}$$

$$b_1 = T[z_1, z_0]$$

$$= \frac{T(z_1) - T(z_0)}{z_1 - z_0}$$

$$= \frac{11.7 - 9.9}{-8 + 9}$$

$$= 1.8$$

$$b_2 = T[z_2, z_1, z_0]$$

$$= \frac{T[z_2, z_1] - T[z_1, z_0]}{z_2 - z_0}$$

$$T[z_2, z_1] = \frac{T(z_2) - T(z_1)}{z_2 - z_1}$$

$$= \frac{17.6 - 11.7}{-7 + 8}$$

$$= 5.9$$

$$T[z_1, z_0] = 1.8$$

$$b_2 = \frac{T[z_2, z_1] - T[z_1, z_0]}{z_2 - z_0}$$

$$= \frac{5.9 - 1.8}{-7 + 9}$$

$$= 2.05$$

$$b_3 = T[z_3, z_2, z_1, z_0]$$

$$= \frac{T[z_3, z_2, z_1] - T[z_2, z_1, z_0]}{z_3 - z_0}$$

$$T[z_3, z_2, z_1] = \frac{T[z_3, z_2] - T[z_2, z_1]}{z_3 - z_1}$$

$$T[z_3, z_2] = \frac{T(z_3) - T(z_2)}{z_3 - z_2}$$

$$= \frac{18.2 - 17.6}{-6 + 7}$$

$$= 0.6$$

$$T[z_2, z_1] = \frac{T(z_2) - T(z_1)}{z_2 - z_1}$$

$$= \frac{17.6 - 11.7}{-7 + 8}$$

$$= 5.9$$

$$T[z_3, z_2, z_1] = \frac{T[z_3, z_2] - T[z_2, z_1]}{z_3 - z_1}$$

$$= \frac{0.6 - 5.9}{-6 + 8}$$

$$= -2.65$$

$$T[z_2, z_1, z_0] = 2.05$$

$$\begin{aligned} b_3 &= T[z_3, z_2, z_1, z_0] \\ &= \frac{T[z_3, z_2, z_1] - T[z_2, z_1, z_0]}{z_3 - z_0} \\ &= \frac{-2.65 - 2.05}{-6 + 9} \\ &= -1.5667 \end{aligned}$$

Hence

$$\begin{aligned} T(z) &= b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1) + b_3(z - z_0)(z - z_1)(z - z_2) \\ &= 9.9 + 1.8(z + 9) + 2.05(z + 9)(z + 8) - 1.5667(z + 9)(z + 8)(z + 7), \ -9 \le z \le -6 \end{aligned}$$

At $z = -7.5$,

$$\begin{aligned} T(-7.5) &= 9.9 + 1.8(-7.5 + 9) + 2.05(-7.5 + 9)(-7.5 + 8) \\ &\quad - 1.5667(-7.5 + 9)(-7.5 + 8)(-7.5 + 7) \\ &= 14.725 ^\circ\text{C} \end{aligned}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$\begin{aligned} |\in_a| &= \left| \frac{14.725 - 14.138}{14.725} \right| \times 100 \\ &= 3.9898\% \end{aligned}$$

If we expand

$$T(z) = 9.9 + 1.8(z + 9) + 2.05(z + 9)(z + 8) - 1.5667(z + 9)(z + 8)(z + 7), \quad -9 \le z \le -6$$

we get

$$T(z) = -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \quad -9 \le z \le -6$$

This is the same expression as obtained in the direct method.

b) To find the position of the thermocline, we must find the points of inflection of the third order polynomial, given by $\frac{d^2T}{dz^2} = 0$

$$\begin{aligned} T(z) &= 9.9 + 1.8(z + 9) + 2.05(z + 9)(z + 8) - 1.5667(z + 9)(z + 8)(z + 7), \quad -9 \le z \le -6 \\ &= -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \quad -9 \le z \le -6 \end{aligned}$$

$$\frac{dT}{dz} = -262.58 - 71.1\text{z} - 4.7z^2, \quad -9 \le z \le -6$$

$$\frac{d^2T}{dz^2} = -71.1 - 9.4z, \quad -9 \le z \le -6$$

Simply setting this expression equal to zero, we get

$$0 = -71.1 - 9.4z$$

$$z = -7.5638 \text{ m}$$

This answer can be verified due to the fact that it falls within the specified range of the third order polynomial.

INTERPOLATION	
Topic	Newton's Divided Difference Interpolation
Summary	Textbook notes on Newton's divided difference interpolation.
Major	Civil Engineering
Authors	Autar Kaw, Michael Keteltas
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Web Site	http://numericalmethods.eng.usf.edu