

## Chapter 07.05

# Gauss Quadrature Rule of Integration

*After reading this chapter, you should be able to:*

- 1. derive the Gauss quadrature method for integration and be able to use it to solve problems, and*
- 2. use Gauss quadrature method to solve examples of approximate integrals.*

### What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

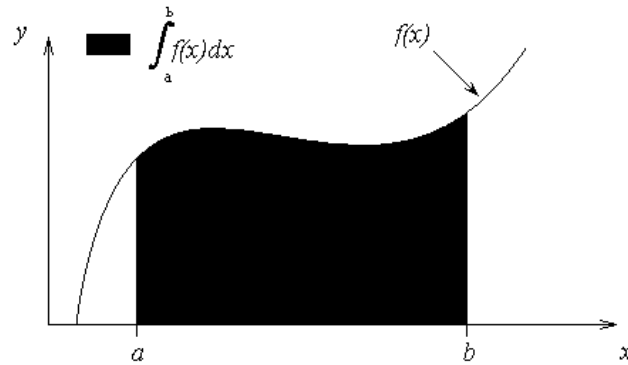
Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral.

Here, we will discuss the Gauss quadrature rule of approximating integrals of the form

$$I = \int_a^b f(x)dx$$

where

- $f(x)$  is called the integrand,
- $a$  = lower limit of integration
- $b$  = upper limit of integration



**Figure 1** Integration of a function.

### Gauss Quadrature Rule

#### Background:

To derive the trapezoidal rule from the method of undetermined coefficients, we approximated

$$\int_a^b f(x) dx \approx c_1 f(a) + c_2 f(b) \quad (1)$$

Let the right hand side be exact for integrals of a straight line, that is, for an integrated form of

$$\int_a^b (a_0 + a_1 x) dx$$

So

$$\begin{aligned} \int_a^b (a_0 + a_1 x) dx &= \left[ a_0 x + a_1 \frac{x^2}{2} \right]_a^b \\ &= a_0 (b - a) + a_1 \left( \frac{b^2 - a^2}{2} \right) \end{aligned} \quad (2)$$

But from Equation (1), we want

$$\int_a^b (a_0 + a_1 x) dx = c_1 f(a) + c_2 f(b) \quad (3)$$

to give the same result as Equation (2) for  $f(x) = a_0 + a_1 x$ .

$$\begin{aligned} \int_a^b (a_0 + a_1 x) dx &= c_1 (a_0 + a_1 a) + c_2 (a_0 + a_1 b) \\ &= a_0 (c_1 + c_2) + a_1 (c_1 a + c_2 b) \end{aligned} \quad (4)$$

Hence from Equations (2) and (4),

$$a_0 (b - a) + a_1 \left( \frac{b^2 - a^2}{2} \right) = a_0 (c_1 + c_2) + a_1 (c_1 a + c_2 b)$$

Since  $a_0$  and  $a_1$  are arbitrary constants for a general straight line

$$c_1 + c_2 = b - a \quad (5a)$$

$$c_1 a + c_2 b = \frac{b^2 - a^2}{2} \quad (5b)$$

Multiplying Equation (5a) by  $a$  and subtracting from Equation (5b) gives

$$c_2 = \frac{b - a}{2} \quad (6a)$$

Substituting the above found value of  $c_2$  in Equation (5a) gives

$$c_1 = \frac{b - a}{2} \quad (6b)$$

Therefore

$$\begin{aligned} \int_a^b f(x) dx &\approx c_1 f(a) + c_2 f(b) \\ &= \frac{b - a}{2} f(a) + \frac{b - a}{2} f(b) \end{aligned} \quad (7)$$

### Derivation of two-point Gauss quadrature rule

#### Method 1:

The two-point Gauss quadrature rule is an extension of the trapezoidal rule approximation where the arguments of the function are not predetermined as  $a$  and  $b$ , but as unknowns  $x_1$  and  $x_2$ . So in the two-point Gauss quadrature rule, the integral is approximated as

$$\begin{aligned} I &= \int_a^b f(x) dx \\ &\approx c_1 f(x_1) + c_2 f(x_2) \end{aligned}$$

There are four unknowns  $x_1$ ,  $x_2$ ,  $c_1$  and  $c_2$ . These are found by assuming that the formula gives exact results for integrating a general third order polynomial,  $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ . Hence

$$\begin{aligned} \int_a^b f(x) dx &= \int_a^b (a_0 + a_1 x + a_2 x^2 + a_3 x^3) dx \\ &= \left[ a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4} \right]_a^b \\ &= a_0 (b - a) + a_1 \left( \frac{b^2 - a^2}{2} \right) + a_2 \left( \frac{b^3 - a^3}{3} \right) + a_3 \left( \frac{b^4 - a^4}{4} \right) \end{aligned} \quad (8)$$

The formula would then give

$$\begin{aligned} \int_a^b f(x) dx &\approx c_1 f(x_1) + c_2 f(x_2) = \\ &c_1 (a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) + c_2 (a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3) \end{aligned} \quad (9)$$

Equating Equations (8) and (9) gives

$$\begin{aligned}
 & a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right) \\
 &= c_1(a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3) + c_2(a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3) \\
 &= a_0(c_1 + c_2) + a_1(c_1x_1 + c_2x_2) + a_2(c_1x_1^2 + c_2x_2^2) + a_3(c_1x_1^3 + c_2x_2^3)
 \end{aligned} \tag{10}$$

Since in Equation (10), the constants  $a_0$ ,  $a_1$ ,  $a_2$ , and  $a_3$  are arbitrary, the coefficients of  $a_0$ ,  $a_1$ ,  $a_2$ , and  $a_3$  are equal. This gives us four equations as follows.

$$\begin{aligned}
 b-a &= c_1 + c_2 \\
 \frac{b^2-a^2}{2} &= c_1x_1 + c_2x_2 \\
 \frac{b^3-a^3}{3} &= c_1x_1^2 + c_2x_2^2 \\
 \frac{b^4-a^4}{4} &= c_1x_1^3 + c_2x_2^3
 \end{aligned} \tag{11}$$

Without proof (see Example 1 for proof of a related problem), we can find that the above four simultaneous nonlinear equations have only one acceptable solution

$$\begin{aligned}
 c_1 &= \frac{b-a}{2} \\
 c_2 &= \frac{b-a}{2} \\
 x_1 &= \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2} \\
 x_2 &= \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}
 \end{aligned} \tag{12}$$

Hence

$$\begin{aligned}
 \int_a^b f(x)dx &\approx c_1f(x_1) + c_2f(x_2) \\
 &= \frac{b-a}{2}f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2}f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right)
 \end{aligned} \tag{13}$$

Method 2:

We can derive the same formula by assuming that the expression gives exact values for the individual integrals of  $\int_a^b 1dx$ ,  $\int_a^b xdx$ ,  $\int_a^b x^2dx$ , and  $\int_a^b x^3dx$ . The reason the formula can also be

derived using this method is that the linear combination of the above integrands is a general third order polynomial given by  $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ .

These will give four equations as follows

$$\begin{aligned}\int_a^b 1 dx &= b - a = c_1 + c_2 \\ \int_a^b x dx &= \frac{b^2 - a^2}{2} = c_1x_1 + c_2x_2 \\ \int_a^b x^2 dx &= \frac{b^3 - a^3}{3} = c_1x_1^2 + c_2x_2^2 \\ \int_a^b x^3 dx &= \frac{b^4 - a^4}{4} = c_1x_1^3 + c_2x_2^3\end{aligned}\quad (14)$$

These four simultaneous nonlinear equations can be solved to give a single acceptable solution

$$\begin{aligned}c_1 &= \frac{b-a}{2} \\ c_2 &= \frac{b-a}{2} \\ x_1 &= \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2} \\ x_2 &= \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\end{aligned}\quad (15)$$

Hence

$$\int_a^b f(x) dx \approx \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right)\quad (16)$$

Since two points are chosen, it is called the two-point Gauss quadrature rule. Higher point versions can also be developed.

### Higher point Gauss quadrature formulas

For example

$$\int_a^b f(x) dx \approx c_1f(x_1) + c_2f(x_2) + c_3f(x_3)\quad (17)$$

is called the three-point Gauss quadrature rule. The coefficients  $c_1$ ,  $c_2$  and  $c_3$ , and the function arguments  $x_1$ ,  $x_2$  and  $x_3$  are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial

$$\int_a^b (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5) dx.$$

General  $n$ -point rules would approximate the integral

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \dots + c_n f(x_n) \quad (18)$$

### Arguments and weighing factors for $n$ -point Gauss quadrature rules

In handbooks (see Table 1), coefficients and arguments given for  $n$ -point Gauss quadrature rule are given for integrals of the form

$$\int_{-1}^1 g(x)dx \approx \sum_{i=1}^n c_i g(x_i) \quad (19)$$

**Table 1** Weighting factors  $c$  and function arguments  $x$  used in Gauss quadrature formulas

Points	Weighting Factors	Function Arguments
2	$c_1 = 1.000000000$	$x_1 = -0.577350269$
	$c_2 = 1.000000000$	$x_2 = 0.577350269$
3	$c_1 = 0.555555556$	$x_1 = -0.774596669$
	$c_2 = 0.888888889$	$x_2 = 0.000000000$
	$c_3 = 0.555555556$	$x_3 = 0.774596669$
4	$c_1 = 0.347854845$	$x_1 = -0.861136312$
	$c_2 = 0.652145155$	$x_2 = -0.339981044$
	$c_3 = 0.652145155$	$x_3 = 0.339981044$
	$c_4 = 0.347854845$	$x_4 = 0.861136312$
5	$c_1 = 0.236926885$	$x_1 = -0.906179846$
	$c_2 = 0.478628670$	$x_2 = -0.538469310$
	$c_3 = 0.568888889$	$x_3 = 0.000000000$
	$c_4 = 0.478628670$	$x_4 = 0.538469310$
	$c_5 = 0.236926885$	$x_5 = 0.906179846$
6	$c_1 = 0.171324492$	$x_1 = -0.932469514$
	$c_2 = 0.360761573$	$x_2 = -0.661209386$
	$c_3 = 0.467913935$	$x_3 = -0.238619186$
	$c_4 = 0.467913935$	$x_4 = 0.238619186$

$c_5 = 0.360761573$	$x_5 = 0.661209386$
$c_6 = 0.171324492$	$x_6 = 0.932469514$

So if the table is given for  $\int_{-1}^1 g(x)dx$  integrals, how does one solve  $\int_a^b f(x)dx$  ?

The answer lies in that any integral with limits of  $[a, b]$  can be converted into an integral with limits  $[-1, 1]$ . Let

$$x = mt + c \quad (20)$$

If  $x = a$ , then  $t = -1$

If  $x = b$ , then  $t = +1$

such that

$$a = m(-1) + c$$

$$b = m(1) + c \quad (21)$$

Solving the two Equations (21) simultaneously gives

$$m = \frac{b-a}{2}$$

$$c = \frac{b+a}{2} \quad (22)$$

Hence

$$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

$$dx = \frac{b-a}{2} dt$$

Substituting our values of  $x$  and  $dx$  into the integral gives us

$$\int_a^b f(x)dx = \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) \frac{b-a}{2} dx \quad (23)$$

### Example 1

For an integral  $\int_{-1}^1 f(x)dx$ , show that the two-point Gauss quadrature rule approximates to

$$\int_{-1}^1 f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

where

$$c_1 = 1$$

$$c_2 = 1$$

$$x_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = \frac{1}{\sqrt{3}}$$

**Solution**

Assuming the formula

$$\int_{-1}^1 f(x) dx = c_1 f(x_1) + c_2 f(x_2) \quad (\text{E1.1})$$

gives exact values for integrals  $\int_{-1}^1 1 dx$ ,  $\int_{-1}^1 x dx$ ,  $\int_{-1}^1 x^2 dx$ , and  $\int_{-1}^1 x^3 dx$ . Then

$$\int_{-1}^1 1 dx = 2 = c_1 + c_2 \quad (\text{E1.2})$$

$$\int_{-1}^1 x dx = 0 = c_1 x_1 + c_2 x_2 \quad (\text{E1.3})$$

$$\int_{-1}^1 x^2 dx = \frac{2}{3} = c_1 x_1^2 + c_2 x_2^2 \quad (\text{E1.4})$$

$$\int_{-1}^1 x^3 dx = 0 = c_1 x_1^3 + c_2 x_2^3 \quad (\text{E1.5})$$

Multiplying Equation (E1.3) by  $x_1^2$  and subtracting from Equation (E1.5) gives

$$c_2 x_2 (x_1^2 - x_2^2) = 0 \quad (\text{E1.6})$$

The solution to the above equation is

$$c_2 = 0, \text{ or/and}$$

$$x_2 = 0, \text{ or/and}$$

$$x_1 = x_2, \text{ or/and}$$

$$x_1 = -x_2.$$

- I.  $c_2 = 0$  is not acceptable as Equations (E1.2-E1.5) reduce to  $c_1 = 2$ ,  $c_1 x_1 = 0$ ,  $c_1 x_1^2 = \frac{2}{3}$ , and  $c_1 x_1^3 = 0$ . But since  $c_1 = 2$ , then  $x_1 = 0$  from  $c_1 x_1 = 0$ , but  $x_1 = 0$  conflicts with  $c_1 x_1^2 = \frac{2}{3}$ .
- II.  $x_2 = 0$  is not acceptable as Equations (E1.2-E1.5) reduce to  $c_1 + c_2 = 2$ ,  $c_1 x_1 = 0$ ,  $c_1 x_1^2 = \frac{2}{3}$ , and  $c_1 x_1^3 = 0$ . Since  $c_1 x_1 = 0$ , then  $c_1$  or  $x_1$  has to be zero but this violates  $c_1 x_1^2 = \frac{2}{3} \neq 0$ .
- III.  $x_1 = x_2$  is not acceptable as Equations (E1.2-E1.5) reduce to  $c_1 + c_2 = 2$ ,  $c_1 x_1 + c_2 x_1 = 0$ ,  $c_1 x_1^2 + c_2 x_1^2 = \frac{2}{3}$ , and  $c_1 x_1^3 + c_2 x_1^3 = 0$ . If  $x_1 \neq 0$ , then  $c_1 x_1 + c_2 x_1 = 0$

gives  $c_1 + c_2 = 0$  and that violates  $c_1 + c_2 = 2$ . If  $x_1 = 0$ , then that violates  $c_1 x_1^2 + c_2 x_1^2 = \frac{2}{3} \neq 0$ .

That leaves the solution of  $x_1 = -x_2$  as the only possible acceptable solution and in fact, it does not have violations (see it for yourself)

$$x_1 = -x_2 \quad (\text{E1.7})$$

Substituting (E1.7) in Equation (E1.3) gives

$$c_1 = c_2 \quad (\text{E1.8})$$

From Equations (E1.2) and (E1.8),

$$c_1 = c_2 = 1 \quad (\text{E1.9})$$

Equations (E1.4) and (E1.9) gives

$$x_1^2 + x_2^2 = \frac{2}{3} \quad (\text{E1.10})$$

Since Equation (E1.7) requires that the two results be of opposite sign, we get

$$x_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = \frac{1}{\sqrt{3}}$$

Hence

$$\begin{aligned} \int_{-1}^1 f(x) dx &= c_1 f(x_1) + c_2 f(x_2) \\ &= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \end{aligned} \quad (\text{E1.11})$$

### Example 2

For an integral  $\int_a^b f(x) dx$ , derive the one-point Gauss quadrature rule.

#### Solution

The one-point Gauss quadrature rule is

$$\int_a^b f(x) dx \approx c_1 f(x_1) \quad (\text{E2.1})$$

Assuming the formula gives exact values for integrals  $\int_{-1}^1 1 dx$ , and  $\int_{-1}^1 x dx$

$$\begin{aligned} \int_a^b 1 dx &= b - a = c_1 \\ \int_a^b x dx &= \frac{b^2 - a^2}{2} = c_1 x_1 \end{aligned} \quad (\text{E2.2})$$

Since  $c_1 = b - a$ , the other equation becomes

$$\begin{aligned}(b-a)x_1 &= \frac{b^2 - a^2}{2} \\ x_1 &= \frac{b+a}{2}\end{aligned}\tag{E2.3}$$

Therefore, one-point Gauss quadrature rule can be expressed as

$$\int_a^b f(x)dx \approx (b-a)f\left(\frac{b+a}{2}\right)\tag{E2.4}$$

### Example 3

What would be the formula for

$$\int_a^b f(x)dx = c_1 f(a) + c_2 f(b)$$

if you want the above formula to give you exact values of  $\int_a^b (a_0 x + b_0 x^2)dx$ , that is, a linear combination of  $x$  and  $x^2$ .

#### Solution

If the formula is exact for a linear combination of  $x$  and  $x^2$ , then

$$\begin{aligned}\int_a^b x dx &= \frac{b^2 - a^2}{2} = c_1 a + c_2 b \\ \int_a^b x^2 dx &= \frac{b^3 - a^3}{3} = c_1 a^2 + c_2 b^2\end{aligned}\tag{E3.1}$$

Solving the two Equations (E3.1) simultaneously gives

$$\begin{aligned}\begin{bmatrix} a & b \\ a^2 & b^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} \frac{b^2 - a^2}{2} \\ \frac{b^3 - a^3}{3} \end{bmatrix} \\ c_1 &= -\frac{1 - ab - b^2 + 2a^2}{6a} \\ c_2 &= -\frac{1 a^2 + ab - 2b^2}{6b}\end{aligned}\tag{E3.2}$$

So

$$\int_a^b f(x)dx = -\frac{1 - ab - b^2 + 2a^2}{6a} f(a) - \frac{1 a^2 + ab - 2b^2}{6b} f(b)\tag{E3.3}$$

Let us see if the formula works.

Evaluate  $\int_2^5 (2x^2 - 3x)dx$  using Equation (E3.3)

$$\begin{aligned} \int_2^5 (2x^2 - 3x) dx &\approx c_1 f(a) + c_2 f(b) \\ &= -\frac{1 - (2)(5) - 5^2 + 2(2)^2}{6} [2(2)^2 - 3(2)] - \frac{1 \cdot 2^2 + 2(5) - 2(5)^2}{6 \cdot 5} [2(5)^2 - 3(5)] \\ &= 46.5 \end{aligned}$$

The exact value of  $\int_2^5 (2x^2 - 3x) dx$  is given by

$$\begin{aligned} \int_2^5 (2x^2 - 3x) dx &= \left[ \frac{2x^3}{3} - \frac{3x^2}{2} \right]_2^5 \\ &= 46.5 \end{aligned}$$

Any surprises?

Now evaluate  $\int_2^5 3dx$  using Equation (E3.3)

$$\begin{aligned} \int_2^5 3dx &\approx c_1 f(a) + c_2 f(b) \\ &= -\frac{1 - 2(5) - 5^2 + 2(2)^2}{6} (3) - \frac{1 \cdot 2^2 + 2(5) - 2(5)^2}{6 \cdot 5} (3) \\ &= 10.35 \end{aligned}$$

The exact value of  $\int_2^5 3dx$  is given by

$$\begin{aligned} \int_2^5 3dx &= [3x]_2^5 \\ &= 9 \end{aligned}$$

Because the formula will only give exact values for linear combinations of  $x$  and  $x^2$ , it does not work exactly even for a simple integral of  $\int_2^5 3dx$ .

Do you see now why we choose  $a_0 + a_1x$  as the integrand for which the formula

$$\int_a^b f(x) dx \approx c_1 f(a) + c_2 f(b)$$

gives us exact values?

#### Example 4

The concentration of benzene at a critical location is given by

$$c = 1.75 \left[ \operatorname{erfc}(0.6560) + e^{32.73} \operatorname{erfc}(5.758) \right]$$

where

$$\operatorname{erfc}(x) = \int_{\infty}^x e^{-z^2} dz$$

So in the above formula

$$\operatorname{erfc}(0.6560) = \int_{\infty}^{0.6560} e^{-z^2} dz$$

Since  $e^{-z^2}$  decays rapidly as  $z \rightarrow \infty$ , we will approximate

$$\operatorname{erfc}(0.6560) = \int_5^{0.6560} e^{-z^2} dz$$

- Use two-point Gauss Quadrature Rule to approximate the value of  $\operatorname{erfc}(0.6560)$ .
- Find the absolute relative true error for part (a).

### Solution

- First, change the limits of integration from  $[5, 0.6560]$  to  $[-1, 1]$  using

$$a = 5$$

$$b = 0.6560$$

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) dx$$

gives

$$\begin{aligned} \int_5^{0.6560} f(z) dz &= \frac{0.6560-5}{2} \int_{-1}^1 f\left(\frac{0.6560-5}{2}z + \frac{0.6560+5}{2}\right) dz \\ &= -2.1720 \int_{-1}^1 f(-2.1720z + 2.8280) dz \end{aligned}$$

Next, get weighting factors and function argument values for the two point rule,

$$c_1 = 1.0000$$

$$z_1 = -0.57735$$

$$c_2 = 1.0000$$

$$z_2 = 0.57735$$

Now we can use the Gauss Quadrature formula

$$\begin{aligned} & -2.1720 \int_{-1}^1 f(-2.1720z + 2.8280) dz \\ & \approx -2.1720 [c_1 f(-2.1720z_1 + 2.8280) + c_2 f(-2.1720z_2 + 2.8280)] \\ & \approx -2.1720 [f(-2.1720(-0.57735) + 2.8280) + f(-2.1720(0.57735) + 2.8280)] \\ & \approx -2.1720 [f(4.0820) + f(1.5740)] \\ & \approx -2.1720 [(5.8003 \times 10^{-8}) + (0.083955)] \\ & \approx -0.18235 \end{aligned}$$

since

$$\begin{aligned} f(4.0820) &= e^{-4.0820^2} \\ &= 5.8003 \times 10^{-8} \end{aligned}$$

$$\begin{aligned} f(1.5740) &= e^{-1.5740^2} \\ &= 0.083955 \end{aligned}$$

b) The absolute relative true error,  $|\epsilon_t|$ , is (Exact value =  $-0.31333$ )

$$\begin{aligned} |\epsilon_t| &= \left| \frac{-0.31333 - (-0.18235)}{-0.31333} \right| \times 100\% \\ &= 41.801\% \end{aligned}$$

### Example 5

The concentration of benzene at a critical location is given by

$$c = 1.75 \left[ \operatorname{erfc}(0.6560) + e^{32.73} \operatorname{erfc}(5.758) \right]$$

where

$$\operatorname{erfc}(x) = \int_{\infty}^x e^{-z^2} dz$$

So in the above formula

$$\operatorname{erfc}(0.6560) = \int_{\infty}^{0.6560} e^{-z^2} dz$$

Since  $e^{-z^2}$  decays rapidly as  $z \rightarrow \infty$ , we will approximate

$$\operatorname{erfc}(0.6560) = \int_5^{0.6560} e^{-z^2} dz$$

- Use three-point Gauss Quadrature Rule to approximate the value of  $\operatorname{erfc}(0.6560)$ .
- Find the absolute relative true error for part (a).

### Solution

a) First, change the limits of integration from  $[5, 0.6560]$  to  $[-1, 1]$  using

$$a = 5$$

$$b = 0.6560$$

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) dx$$

gives

$$\begin{aligned} \int_5^{0.6560} f(z) dz &= \frac{0.6560-5}{2} \int_{-1}^1 f\left(\frac{0.6560-5}{2}z + \frac{0.6560+5}{2}\right) dz \\ &= -2.1720 \int_{-1}^1 f(-2.1720z + 2.8280) dz \end{aligned}$$

The weighting factors and function argument values are

$$c_1 = 0.55556$$

$$z_1 = -0.77460$$

$$\begin{aligned}c_2 &= 0.88889 \\z_2 &= 0.0000 \\c_3 &= 0.55556 \\z_3 &= 0.77460\end{aligned}$$

and the formula is

$$\begin{aligned}& -2.1720 \int_{-1}^1 f(-2.1720z + 2.8280) dz \\& \approx -2.1720 [c_1 f(-2.1720z_1 + 2.8280) + c_2 f(-2.1720z_2 + 2.8280) + c_3 f(-2.1720z_3 + 2.8280)] \\& \approx -2.1720 \left[ \begin{aligned} & 0.55556 f(-2.1720(-0.77460) + 2.8280) \\ & + 0.88889 f(-2.1720(0.0000) + 2.8280) \\ & + 0.55556 f(-2.1720(0.77460) + 2.8280) \end{aligned} \right] \\& \approx -2.1720 [0.55556 f(4.5104) + 0.88889 f(2.8280) + 0.55556 f(1.1456)] \\& \approx -2.1720 [0.55556(1.4616 \times 10^{-9}) + 0.88889(3.3627 \times 10^{-4}) + 0.55556(0.26919)] \\& \approx -0.32547\end{aligned}$$

since

$$\begin{aligned}f(4.5104) &= e^{-4.5104^2} \\&= 1.4616 \times 10^{-9} \\f(2.8280) &= e^{-2.8280^2} \\&= 3.3627 \times 10^{-4} \\f(1.1456) &= e^{-1.1456^2} \\&= 0.26917\end{aligned}$$

b) The absolute relative true error,  $|\epsilon_t|$ , is (Exact value = -0.31333)

$$\begin{aligned}|\epsilon_t| &= \left| \frac{-0.31333 - (-0.32547)}{-0.31333} \right| \times 100\% \\&= 3.8757 \%\end{aligned}$$

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## INTEGRATION

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Topic	Gauss quadrature rule
Summary	These are textbook notes of Gauss quadrature rule
Major	Civil Engineering
Authors	Autar Kaw, Michael Keteltas
Date	August 27, 2009
Web Site	<a href="http://numericalmethods.eng.usf.edu">http://numericalmethods.eng.usf.edu</a>

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