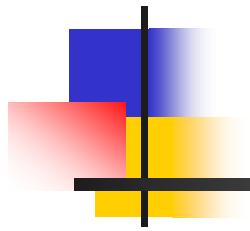


Ordinary Differential Equations



Topic: Euler Method

Major: Civil Engineering

Authors: Autar Kaw, Charlie Barker

Euler's Method

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

$$\begin{aligned} \text{Slope} &= \frac{\text{Rise}}{\text{Run}} \\ &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= f(x_0, y_0) \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)(x_1 - x_0) \\ &= y_0 + f(x_0, y_0)h \end{aligned}$$

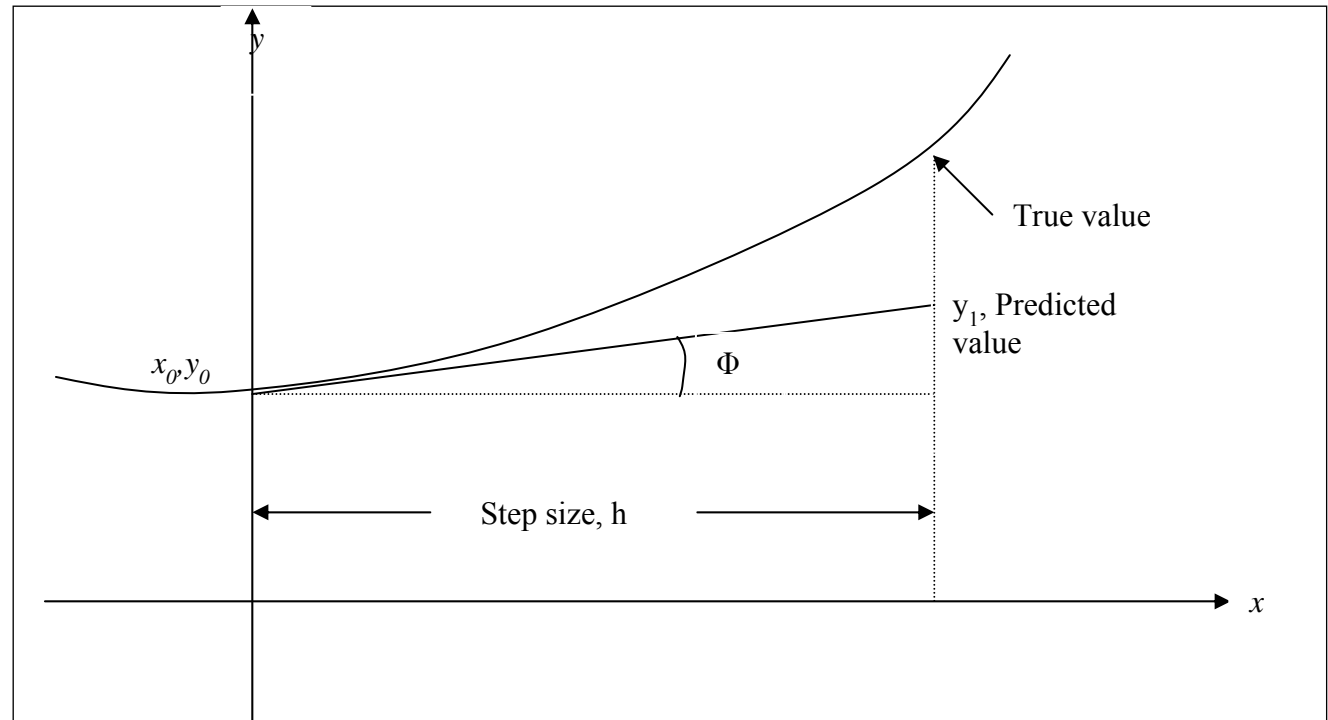


Figure 1. Graphical interpretation of the first step of Euler's method

Euler's Method

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$h = x_{i+1} - x_i$$

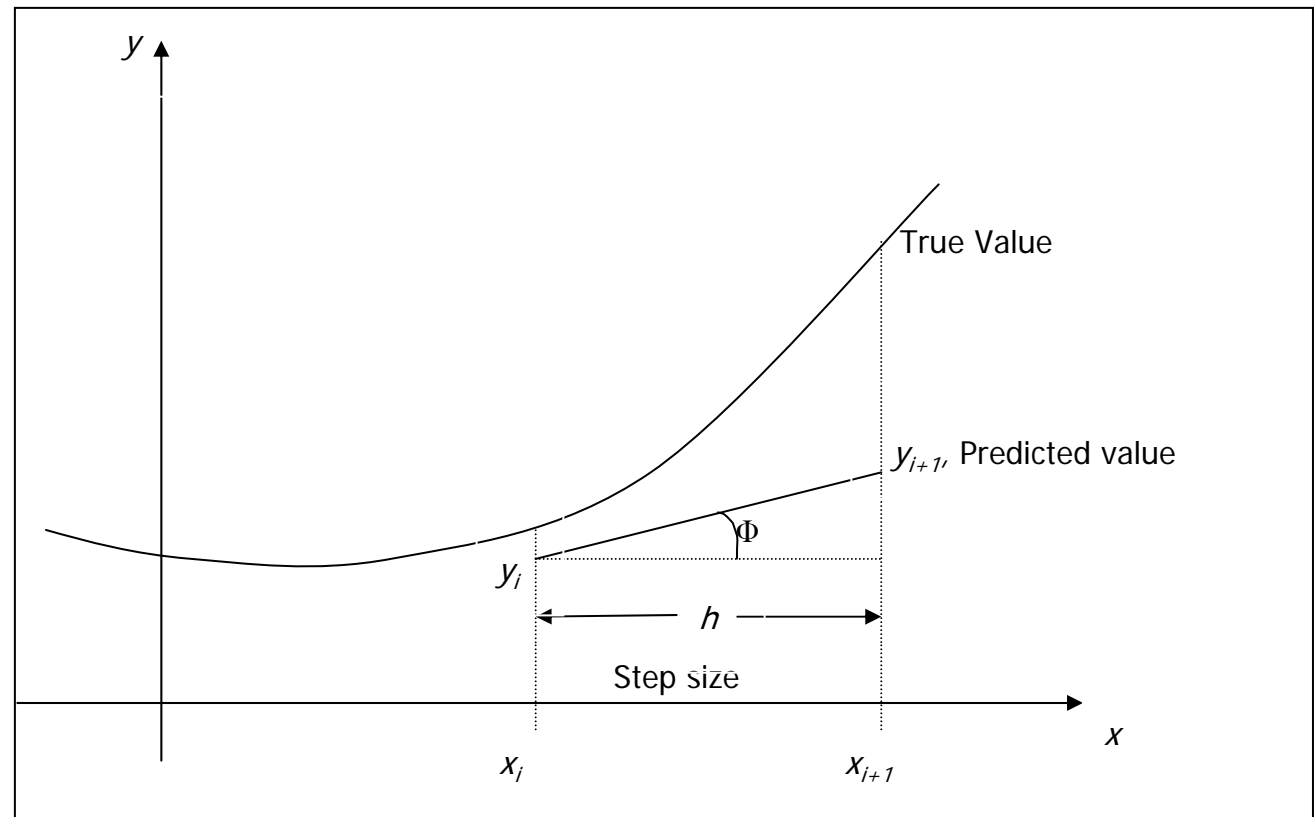


Figure 2. General graphical interpretation of Euler's method



How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$



Example

A polluted lake with an initial concentration of a bacteria is 10^7 parts/m³, while the acceptable level is only 5×10^6 parts/m³. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration C of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt} + 0.06C = 0, C(0) = 10^7$$

Find the concentration of the pollutant after 7 weeks. Take a step size of 3.5 weeks.

$$\frac{dC}{dt} = -0.06C$$

$$f(t, C) = -0.06C$$

$$C_{i+1} = C_i + f(t_i, C_i)h$$



Solution

Step 1:

$$\begin{aligned}C_1 &= C_0 + f(t_0, C_0)h \\&= 10^7 + f(0, 10^7)3.5 \\&= 10^7 + (-0.06(10^7))3.5 \\&= 10^7 + (-6 \times 10^5)3.5 \\&= 7.9000 \times 10^6 \text{ parts} / m^3\end{aligned}$$

C_1 is the approximate concentration of bacteria at

$$t = t_1 = t_0 + h = 0 + 3.5 = 3.5 \text{ weeks}$$

$$C_1 = C(3.5) \cong 7.9000 \times 10^6 \text{ parts} / m^3$$



Solution Cont

Step 2:

$$\begin{aligned}C_2 &= C_1 + f(t_1, C_1)h \\&= 7.9000 \times 10^6 + f(3.5, 7.9000 \times 10^6)3.5 \\&= 7.9000 \times 10^6 + (-0.06(7.9000 \times 10^6))3.5 \\&= 7.9000 \times 10^6 + (-4.7400 \times 10^5)3.5 \\&= 6.2410 \times 10^6\end{aligned}$$

C_2 is the approximate concentration of bacteria at

$$t = t_2 = t_1 + h = 3.5 + 3.5 = 7 \text{ weeks}$$

$$C_2 = C(7) \cong 6.2410 \times 10^6$$



Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$C(t) = 10000000e^{\left(\frac{-3t}{50}\right)}$$

The solution to this nonlinear equation at $t=7$ weeks is

$$C(7) = 6.5705 \times 10^6$$

Comparison of Exact and Numerical Solutions

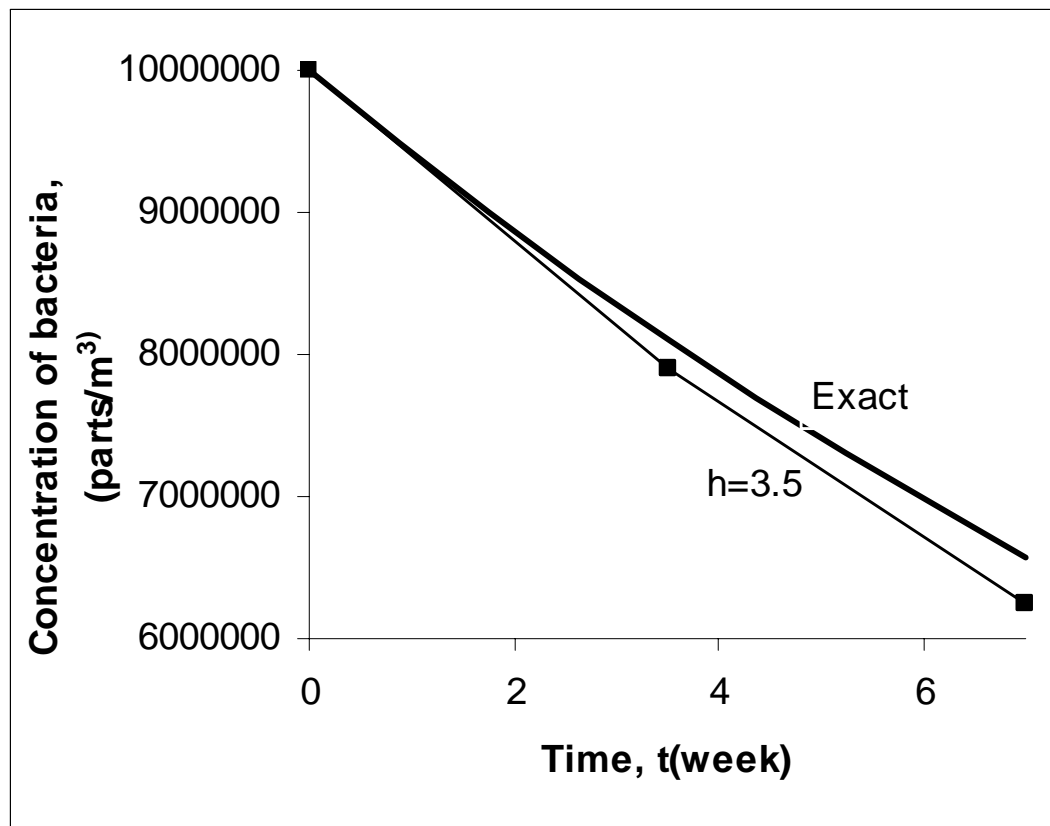
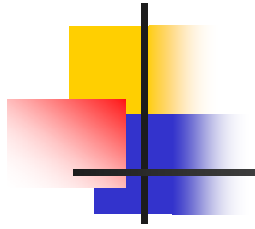


Figure 3. Comparing exact and Euler's method



Effect of step size

Table 1. Concentration of bacteria after 7 weeks as a function of step size

Step h	$C(7)$	E_t	$ \epsilon_t \%$
7	5.8000×10^6	770470	11.726
3.5	6.2410×10^6	329470	5.0144
1.75	6.4160×10^6	154060	2.3447
0.875	6.4960×10^6	74652	1.1362
0.4375	6.5340×10^6	36763	0.55952

Comparison with exact results

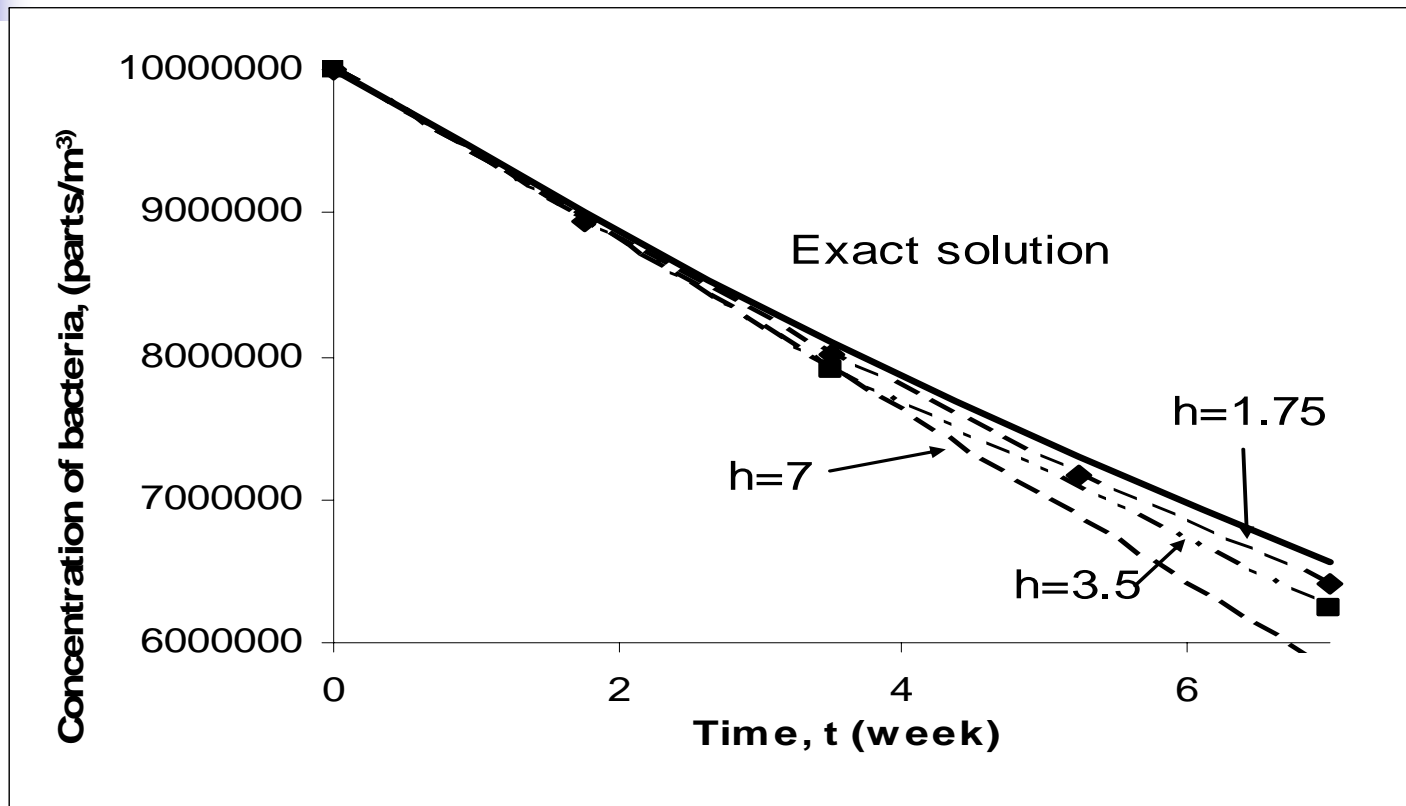


Figure 4. Comparison of Euler's method with exact solution for different step sizes

Effects of step size on Euler's Method

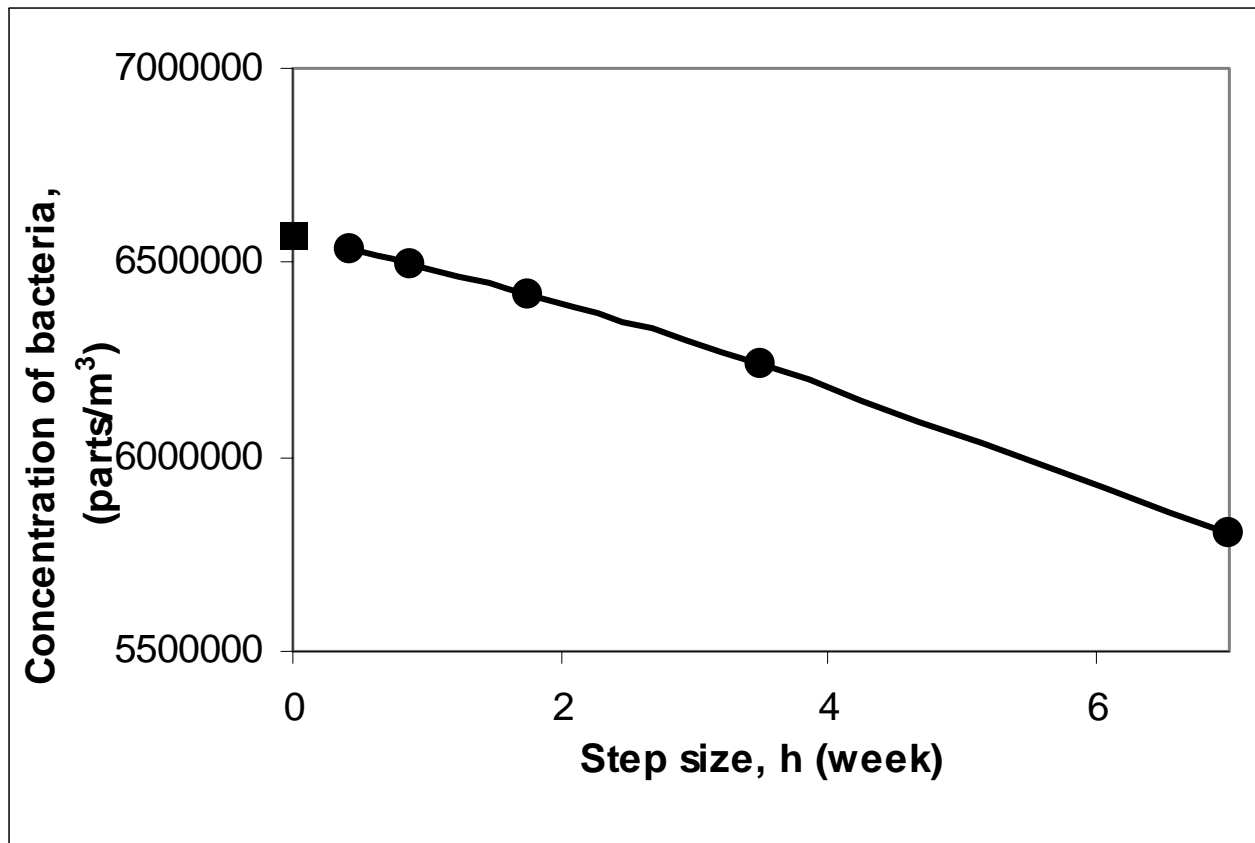


Figure 5. Effect of step size in Euler's method.



Errors in Euler's Method

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \left. \frac{dy}{dx} \right|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!} \left. \frac{d^2 y}{dx^2} \right|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!} \left. \frac{d^3 y}{dx^3} \right|_{x_i, y_i} (x_{i+1} - x_i)^3 + \dots$$

$$y_{i+1} = y_i + f(x_i, y_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \dots$$

As you can see the first two terms of the Taylor series

$$y_{i+1} = y_i + f(x_i, y_i)h \quad \text{are the Euler's method.}$$

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \dots \quad E_t \propto h^2$$