Holistic
Numerical
Methods
Institute

Ordinary Differential Equations

Topic: Runge-Kutta 2nd Order Method

Major: Civil Engineering

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Runge-Kutta 2nd Order Method

For $$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

Runge Kutta 2nd order method is given by

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

Heun’s Method

Heun’s method

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + h, y_i + k_1 h\right)$$

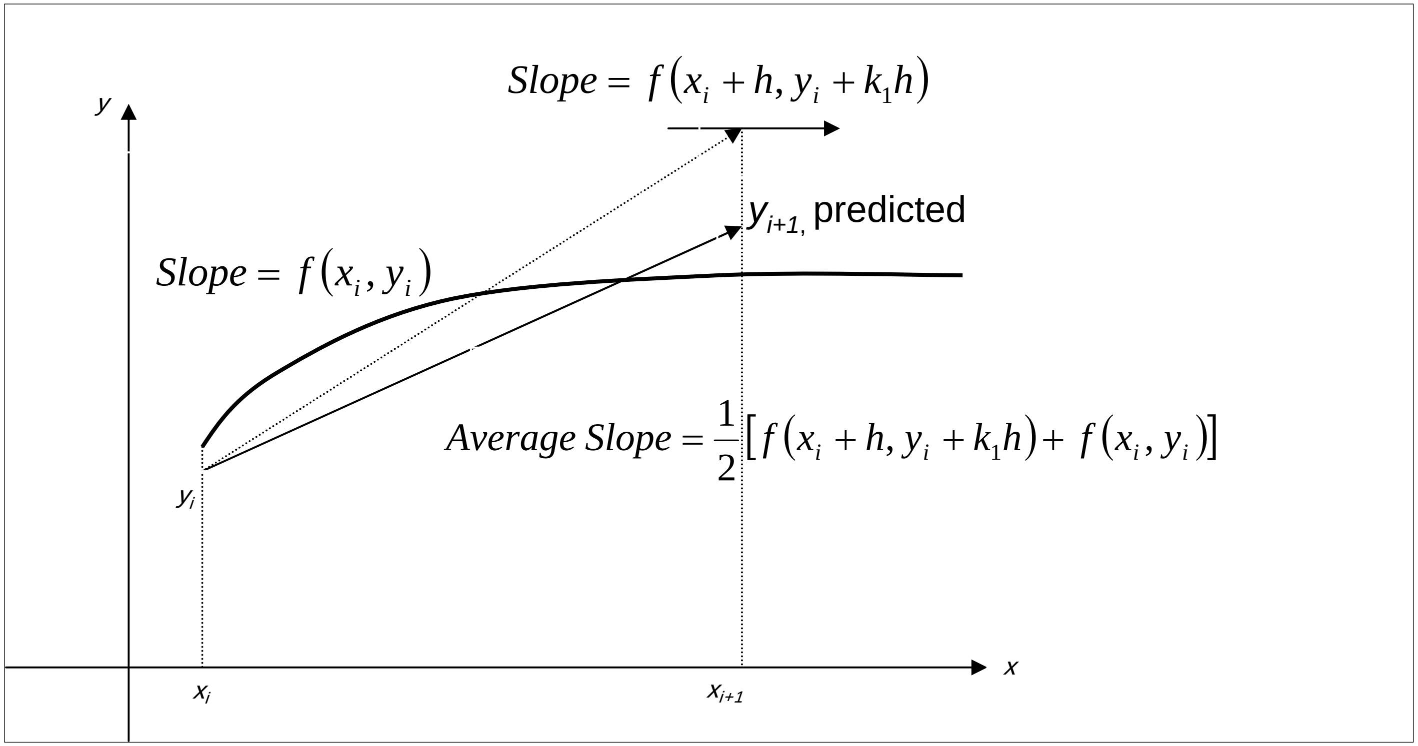

Figure 1. Runge-Kutta 2nd order method (Heun’s method)

Midpoint Method

Here $a_2 = 1$ **is chosen, giving**

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

Ralston's Method

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + (\frac{1}{3}k_1 + \frac{2}{3}k_2)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A polluted lake with an initial concentration of a bacteria is 10^7 parts/m^3, while the acceptable level is only $5\text{x}10^6$ parts/m^3. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration *C* of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt} + 0.06C = 0, C(0) = 10^7$$

Find the concentration of the pollutant after 7 weeks. Take a step size of 3.5 weeks.

$$\frac{dC}{dt} = -0.06C$$

$$f(t, C) = -0.06C$$

$$C_{i+1} = C_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

Solution

Step 1: $i = 0, t_0 = 0, C_0 = C(0) = 10^7$

$$k_1 = f\left(t_0, C_0\right) = f\left(0,10^7\right) = -0.06(10^7) = -600000$$

$$k_2 = f\left(t_0 + h, C_0 + k_1 h\right) = f\left(0 + 3.5, 10^7 + (-600000)3.5\right) = f\left(3.5, 7.9000 \times 10^6\right)$$

$$= -0.06(7.9000 \times 10^6) = -474000$$

$$C_1 = C_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$= 10^7 + \left(\frac{1}{2}(-600000) + \frac{1}{2}(-474000)\right)3.5$$

$$= 10^7 + (-537000)3.5$$

$$= 8.1205 \times 10^6$$

Solution Cont

Step 2: $i=1, t_1 = t_0 + h = 0 + 3.5 = 3.5, C_1 = 8.1205 \times 10^6 \; parts/m^3$

$$k_1 = f\left(t_1, C_1\right) = f\left(3.5{,}8120500\right) = -0.06(8120500) = -487230$$

$$k_2 = f\left(t_1 + h, C_1 + k_1 h\right) = f\left(3.5 + 3.5{,}8120500 + (-487239)3.5\right) = f\left(7{,}6415200\right)$$

$$= -0.06(6415200) = -384910$$

$$C_2 = C_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$= 8120500 + \left(\frac{1}{2}(-487230) + \frac{1}{2}(-384910)\right)3.5$$

$$= 8120500 + (-436070)3.5$$

$$= 6.5943 \times 10^6 \; parts/m^3$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$C(t) = 10000000e^{(\frac{-3t}{50})}$$

The solution to this nonlinear equation at t=7 weeks is

$$C(7) = 6.5705 \times 10^{6}$$

Comparison with exact results

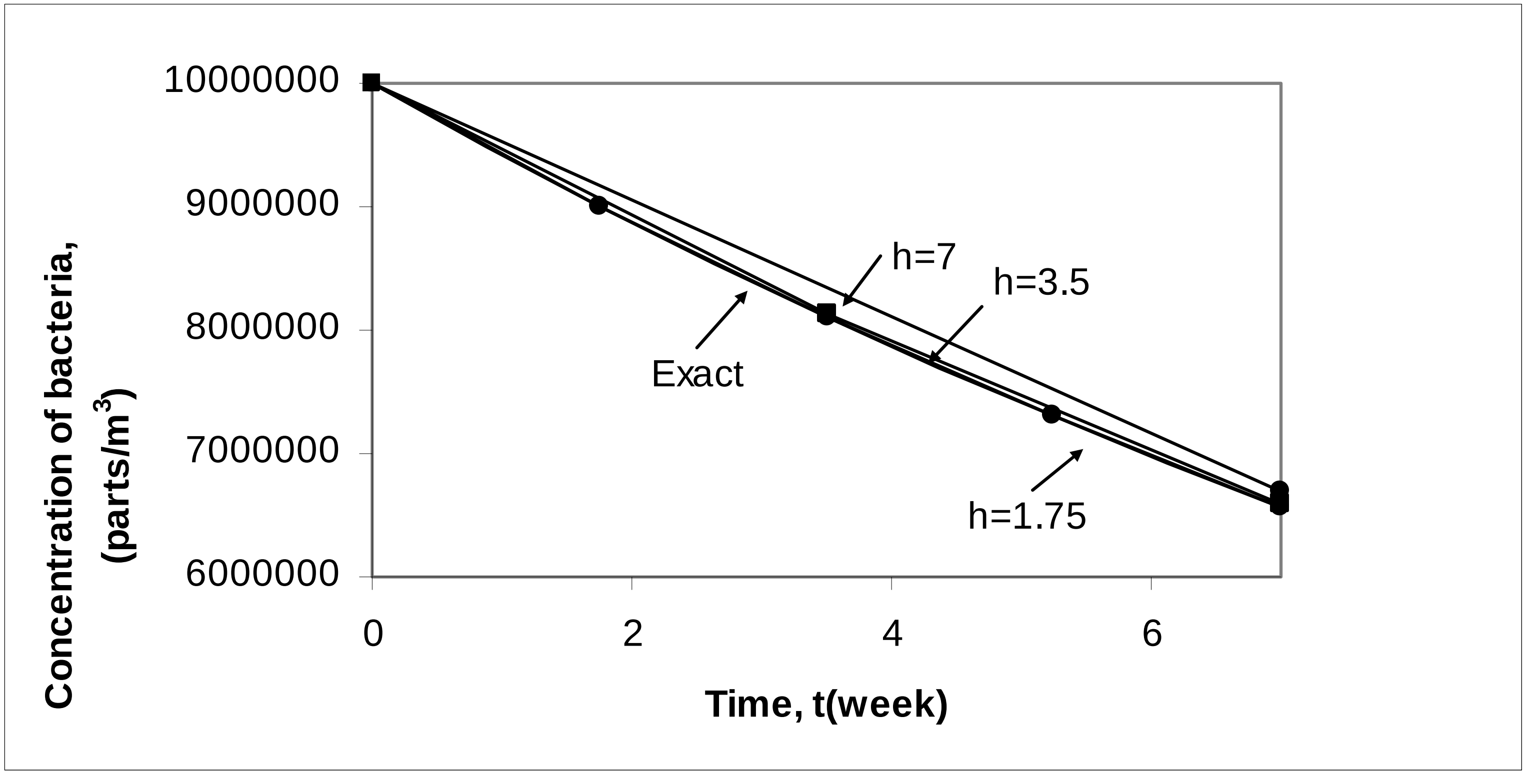

Figure 2. Heun's method results for different step sizes

Effect of step size

Table 1. Effect of step size for Heun's method

Step Size h	$C(7)$	E_t	$\lvert\in_t\rvert\,\%$
7	6.6820×10^{6}	-111530	1.6975
3.5	6.5943×10^{6}	-23784	0.36198
1.75	6.5760×10^{6}	-5489.1	0.083542
0.875	6.5718×10^{6}	-1318.8	0.020071
0.4375	6.5708×10^{6}	-323.24	0.0049195

$C(7) = 6.5705\times10^{6}$ (exact)

Effects of step size on Heun's Method

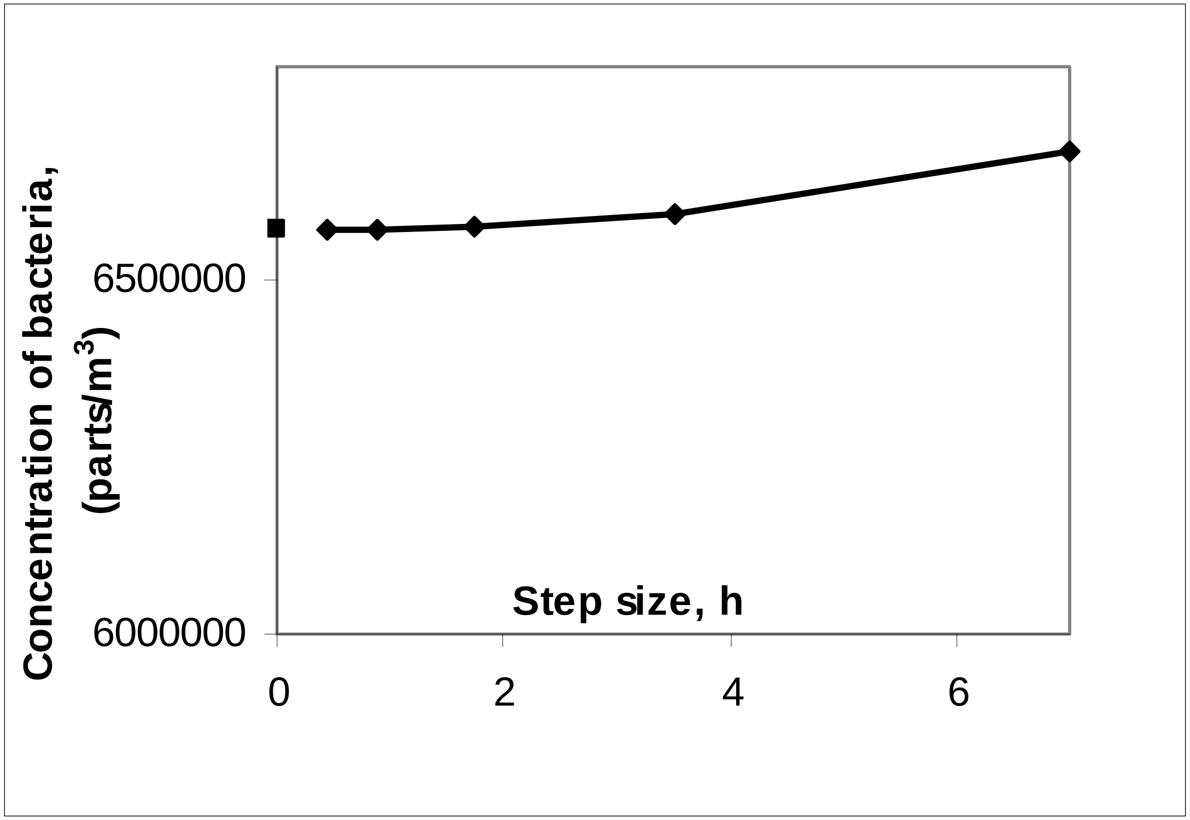

Figure 3. Effect of step size in Heun's method

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$C(7)$			
	Euler	Heun	Midpoint	Ralston
7	5.8000×10^6	6.6820×10^6	6.6820×10^6	6.6820×10^6
3.5	6.2410×10^6	6.5943×10^6	6.5943×10^6	6.5943×10^6
1.75	6.4160×10^6	6.5760×10^6	6.5760×10^6	6.5760×10^6
0.875	6.4960×10^6	6.5718×10^6	6.5718×10^6	6.5718×10^6
0.4375	6.5340×10^6	6.5708×10^6	6.5708×10^6	6.5708×10^6

$C(7) = 6.5705\times10^6$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\left\|\in_t\right\|\%$			
	Euler	Heun	Midpoint	Ralston
7	11.726	1.6975	1.6975	1.6975
3.5	5.0144	0.36198	0.36198	0.36198
1.75	2.3447	0.083542	0.083542	0.083542
0.875	1.1362	0.020071	0.020071	0.020071
0.4375	0.55952	0.0049195	0.0049195	0.0049195

$C(7) = 6.5705 \times 10^6$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

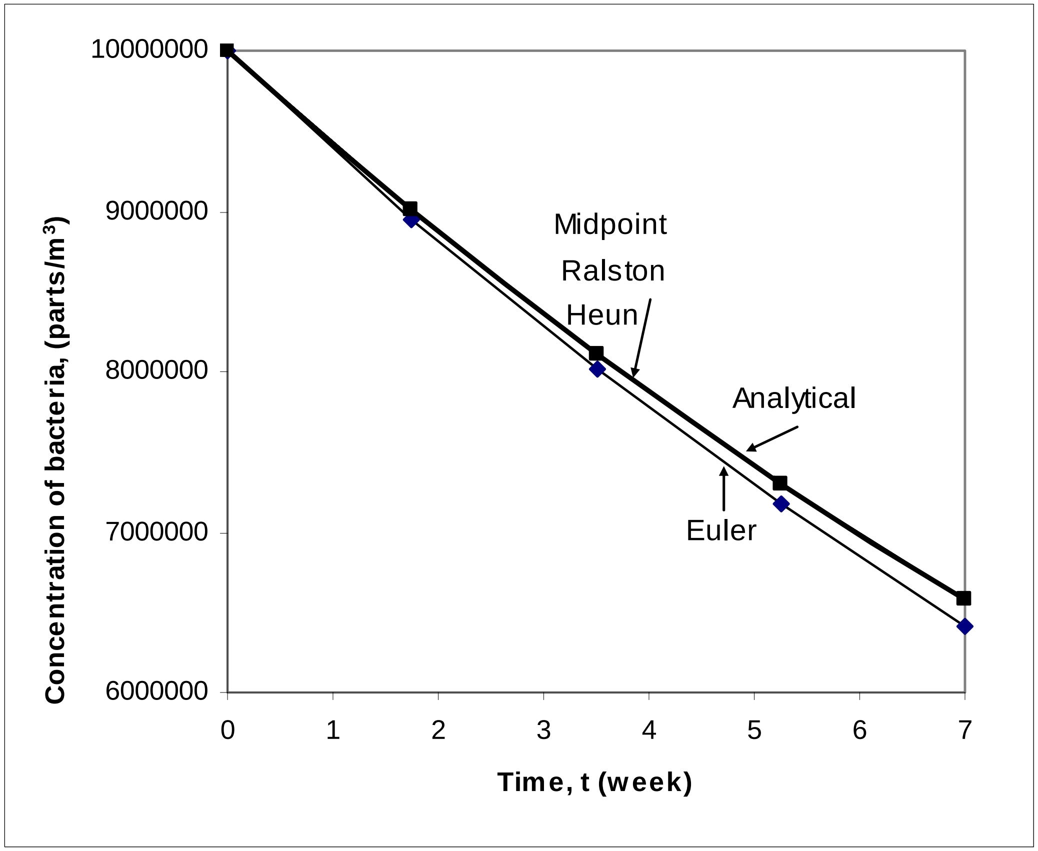

Figure 4. Comparison of Euler and Runge Kutta 2nd order methods with exact results.