Ordinary Differential Equations

Topic: Runge-Kutta 4th Order Method

Major: Civil Engineering

Authors: Autar Kaw, Charlie Barker

Runge-Kutta 4th Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 4th order method is given by

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x,y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x,y) = 1.3e^{-x} - 2y$$

Example

A polluted lake with an initial concentration of a bacteria is 10^7 parts/m^3, while the acceptable level is only 5×10^6 parts/m^3. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration *C* of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt} + 0.06C = 0, C(0) = 10^7$$

Find the concentration of the pollutant after 7 weeks. Take a step size of 3.5 weeks.

$$\frac{dC}{dt} = -0.06C$$

$$f(t,C) = -0.06C$$

$$C_{i+1} = C_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

Solution

Step 1: $i = 0 \quad t_0 = 0 \quad C_0 = 10^7\ parts/m^3$

$$k_1 = f(t_0, C_0) = f(0, 10^7) = -0.06(10^7) = -600000$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, C_0 + \frac{1}{2}k_1 h\right) = f\left(0 + \frac{1}{2}3.5, 10^7 + \frac{1}{2}(-600000)3.5\right) = f(1.75, 8950000)$$

$$= -0.06(8950000) = -537000$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, C_0 + \frac{1}{2}k_2 h\right) = f\left(0 + \frac{1}{2}3.5, 10^7 + \frac{1}{2}(-537000)3.5\right) = f(1.75, 9060300)$$

$$= -0.06(9060300) = -543620$$

$$k_4 = f\left(t_0 + \frac{1}{2}h, C_0 + \frac{1}{2}k_3 h\right) = f\left(0 + \frac{1}{2}3.5, 10^7 + \frac{1}{2}(-543620)3.5\right) = f(1.75, 9048700)$$

$$= -0.06(9048700) = -542920$$

Solution Cont

$$C_1 = C_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 10^7 + \frac{1}{6}\left(-600000 + 2\left(-537000\right) + 2\left(-543620\right) + \left(-542920\right)\right)3.5$$

$$= 10^7 + \frac{1}{6}\left(-3304200\right)3.5$$

$$= 8.0726 \times 10^6$$

C_1 is the approximate concentration of bacteria at

$$t = t_1 = t_0 + h = 0 + 3.5 = 3.5$$

$$C_1 = C\left(3.5\right) \approx 8.0726 \times 10^6$$

Solution Cont

Step 2: $i = 1, t_1 = 3.5, C_1 = 8.0726 \times 10^6$

$$k_1 = f(t_1, C_1) = f\left(3.5, 8.0726 \times 10^6\right) = -0.06(8.0726 \times 10^6) = -484350$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, C_1 + \frac{1}{2}k_1 h\right) = f\left(3.5 + \frac{1}{2}3.5, 8072600 + \frac{1}{2}(-484350)3.5\right)$$

$$= f(5.25, 7225000) = -0.06(7225000) = -433500$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, C_1 + \frac{1}{2}k_2 h\right) = f\left(3.5 + \frac{1}{2}3.5, 8072600 + \frac{1}{2}(-433500)3.5\right)$$

$$= f(5.25, 7314000) = -0.06(7314000) = -438840$$

$$k_4 = f\left(t_1 + \frac{1}{2}h, C_1 + k_3 h\right) = f\left(3.5 + \frac{1}{2}3.5, 8072600 + \frac{1}{2}(-438840)3.5\right)$$

$$= f(5.25, 7304600) = -0.06(7304600) = -438280$$

Solution Cont

$$C_2 = C_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 8072600 + \frac{1}{6}\left(-484350 + 2\left(-433500\right) + 2\left(-438840\right) + (-438280)\right)3.5$$

$$= 8072600 + \frac{1}{6}\left(-2667300\right)3.5$$

$$= 6.5167 \times 10^6$$

C_2 is the approximate concentration of bacteria at

$t_2 = t_1 + h$ = $3.5 + 3.5 = 7\,weeks$

$$C_2 = C\left(7\right) \approx 6.5167 \times 10^6$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$C(t) = 10000000e^{(\frac{-3t}{50})}$$

The solution to this nonlinear equation at t=7 weeks is

$$C(7) = 6.5705 \times 10^{6}$$

Comparison with exact results

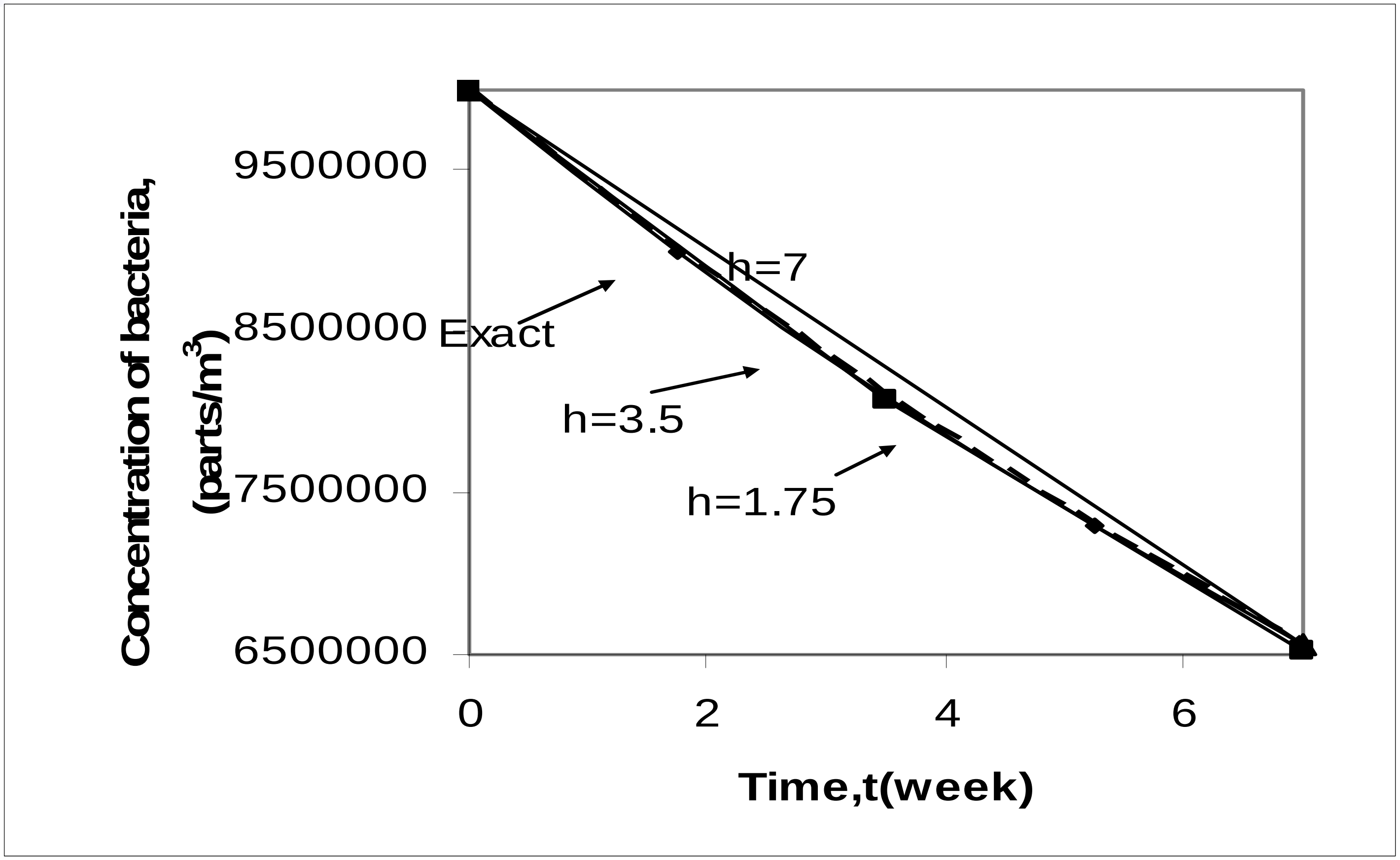

Figure 1. Comparison of Runge-Kutta 4th order method with exact solution

Effect of step size

Table 1. Value of concentration of bacteria at 3 minutes for different step sizes

Step h	$x(3)$	E_t	$\lvert \in_t \rvert$ %
7	6.5715×10^6	-1017.2	0.015481
3.5	6.5705×10^6	0	0
1.75	6.5705×10^6	0	0
0.875	6.5705×10^6	0	0
0.4375	6.5705×10^6	0	0

$$C(7) = 6.5705\times10^6 \quad \text{(exact)}$$

Effects of step size on Runge-Kutta 4th Order Method

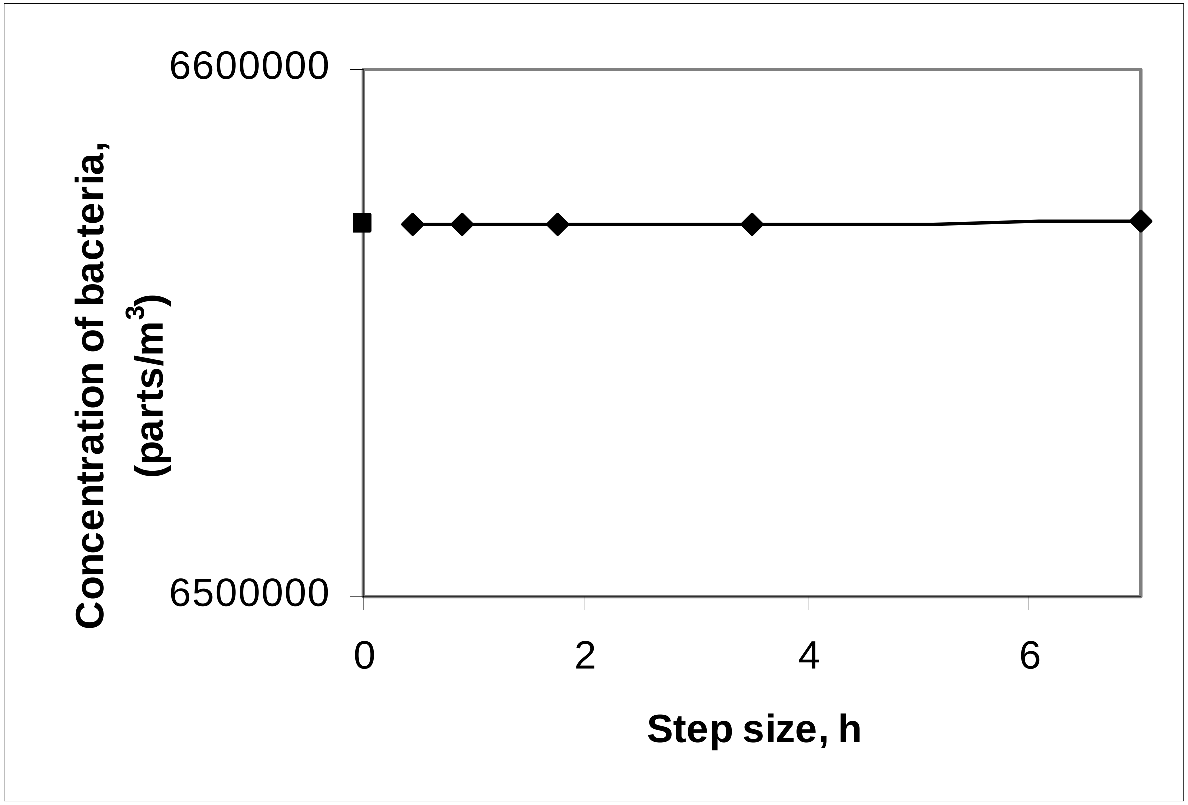

Figure 2. Effect of step size in Runge-Kutta 4th order method

Comparison of Euler and Runge-Kutta Methods

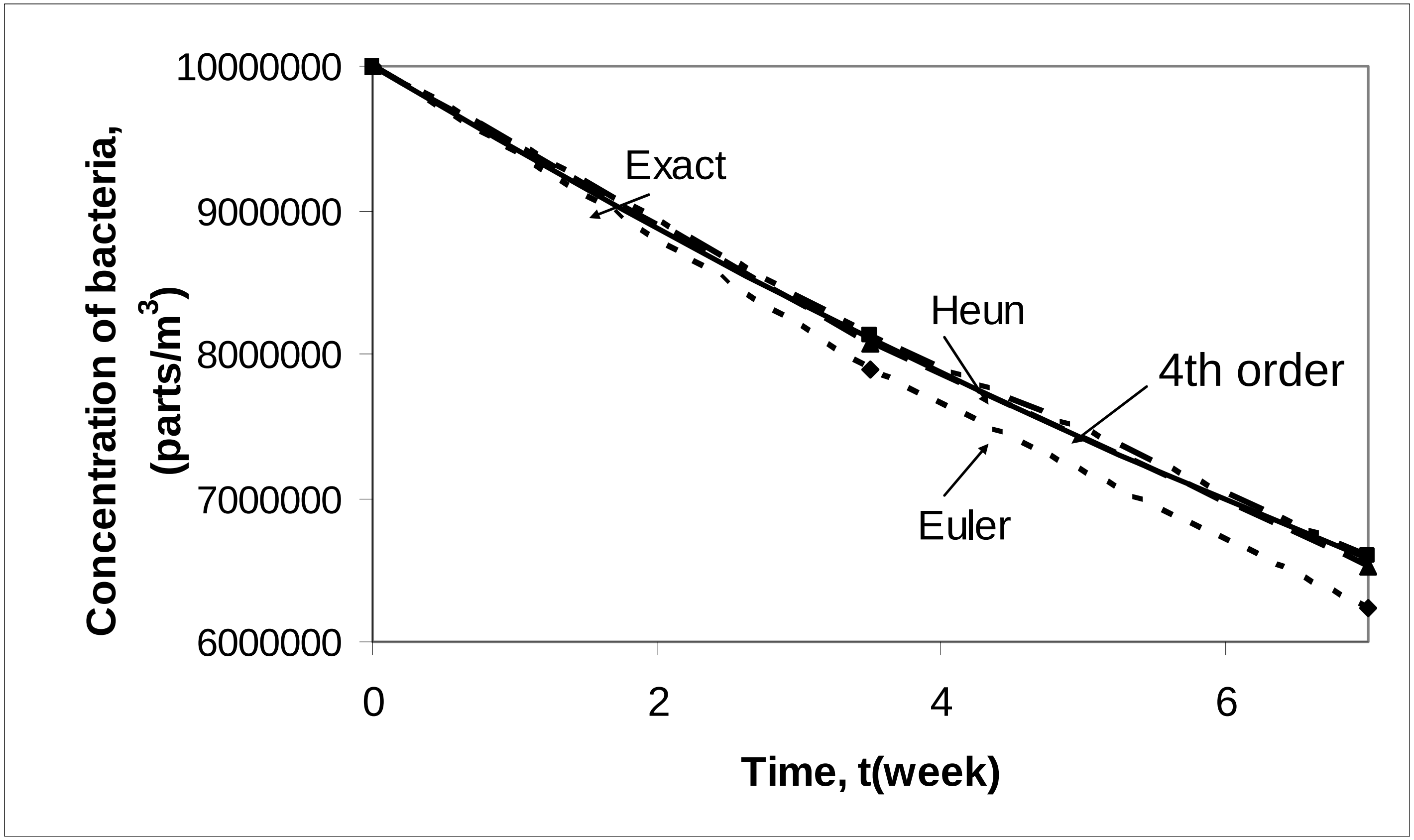

Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.