Chapter 08.02
Euler's Method for Ordinary Differential Equations

After reading this chapter, you should be able to:

1. *develop Euler's Method for solving ordinary differential equations,*
2. *determine how the step size affects the accuracy of a solution,*
3. *derive Euler's formula from Taylor series, and*
4. *use Euler's method to find approximate values of integrals.*

What is Euler's method?

Euler's method is a numerical technique to solve ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0 \quad (1)$$

So only first order ordinary differential equations can be solved by using Euler's method. In another chapter we will discuss how Euler's method is used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations. How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \ y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x,y) = 1.3e^{-x} - 2y$$

Example 2

Rewrite

$$e^{y}\frac{dy}{dx} + x^2y^2 = 2\sin(3x),\ \ y(0) = 5$$

in

$$\frac{dy}{dx} = f(x,y),\ \ y(0) = y_0 \text{ form.}$$

Solution

$$e^{y}\frac{dy}{dx} + x^2y^2 = 2\sin(3x),\ \ y(0) = 5$$

$$\frac{dy}{dx} = \frac{2\sin(3x) - x^2y^2}{e^{y}},\ \ y(0) = 5$$

In this case

$$f(x,y) = \frac{2\sin(3x) - x^2y^2}{e^{y}}$$

Derivation of Euler's method

At $x = 0$, we are given the value of $y = y_0$. Let us call $x = 0$ as x_0. Now since we know the slope of y with respect to x, that is, $f(x,y)$, then at $x = x_0$, the slope is $f(x_0, y_0)$. Both x_0 and y_0 are known from the initial condition $y(x_0) = y_0$.

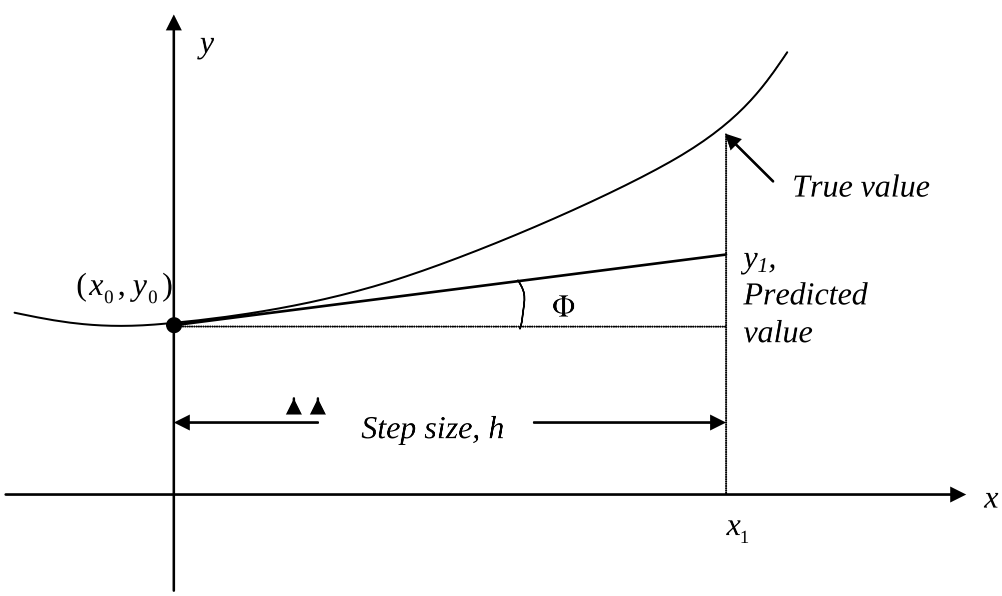

Figure 1 Graphical interpretation of the first step of Euler's method.

So the slope at $x = x_0$ as shown in Figure 1 is

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

From here

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

Calling $x_1 - x_0$ the step size h, we get

$$y_1 = y_0 + f(x_0, y_0)h \qquad (2)$$

One can now use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2, and that would be the predicted value at x_2, given by

$$y_2 = y_1 + f(x_1, y_1)h$$

$$x_2 = x_1 + h$$

Based on the above equations, if we now know the value of $y = y_i$ at x_i, then

$$y_{i+1} = y_i + f(x_i, y_i)h \qquad (3)$$

This formula is known as Euler's method and is illustrated graphically in Figure 2. In some books, it is also called the Euler-Cauchy method.

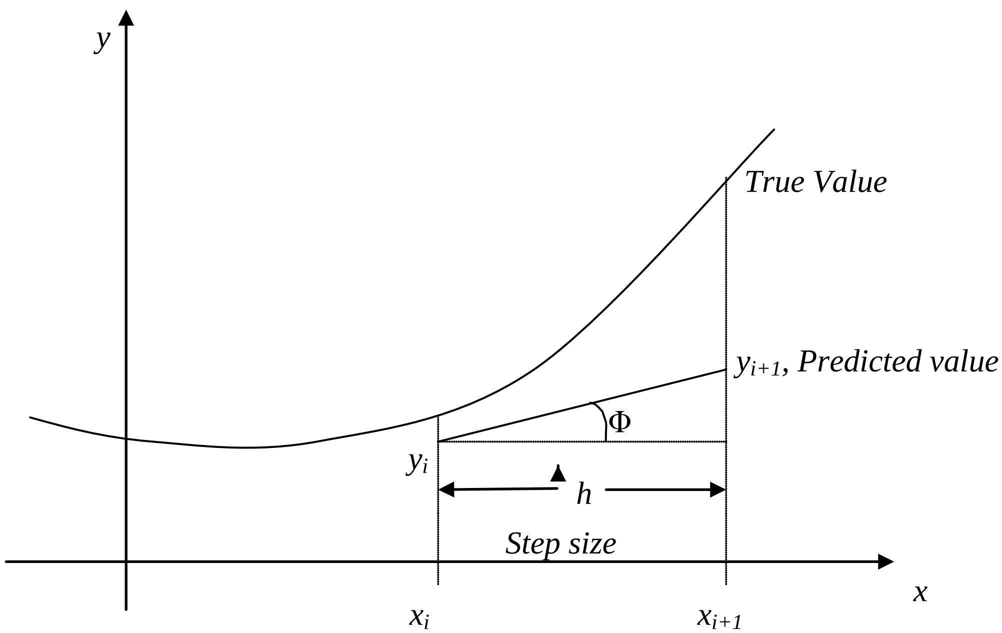

Figure 2 General graphical interpretation of Euler's method.

Example 3

A polluted lake has an initial concentration of a bacteria of 10^7 parts/m^3, while the acceptable level is only 5×10^6 parts/m^3. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration C of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt}+0.06C=0,\ C(0)=10^7$$

Using Euler's method and a step size of 3.5 weeks, find the concentration of the pollutant after 7 weeks.

Solution

$$\frac{dC}{dt}=-0.06C$$

$$f(t,C)=-0.06C$$

The Euler's method reduces to

$$C_{i+1}=C_i+f(t_i,C_i)h$$

For $i=0$, $t_0=0$, $C_0=10^7$

$$\begin{aligned}C_1&=C_0+f(t_0,C_0)h\\&=10^7+f(0,10^7)3.5\\&=10^7+\left(-0.06(10^7)\right)3.5\\&=10^7+\left(-6\times10^5\right)3.5\\&=7.9\times10^6\text{parts/m}^3\end{aligned}$$

C_1 is the approximate concentration of bacteria at

$$t=t_1=t_0+h=0+3.5=3.5\text{weeks}$$

$$C(3.5)\approx C_1=7.9\times10^6\text{ parts/m}^3$$

For $i=1$, $t_1=3.5$, $C_1=7.9\times10^6$

$$\begin{aligned}C_2&=C_1+f(t_1,C_1)h\\&=7.9\times10^6+f\left(3.5,7.9\times10^6\right)3.5\\&=7.9\times10^6+\left(-0.06(7.9\times10^6)\right)3.5\\&=7.9\times10^6+\left(-4.74\times10^5\right)3.5\\&=6.241\times10^6\text{ parts/m}^3\end{aligned}$$

C_2 is the approximate concentration of bacteria at

$$t=t_2=t_1+h=3.5+3.5=7\text{weeks}$$

$$C(7)\approx C_2=6.241\times10^6\text{ parts/m}^3$$

Figure 3 compares the exact solution with the numerical solution from Euler's method for the step size of $h=3.5$.

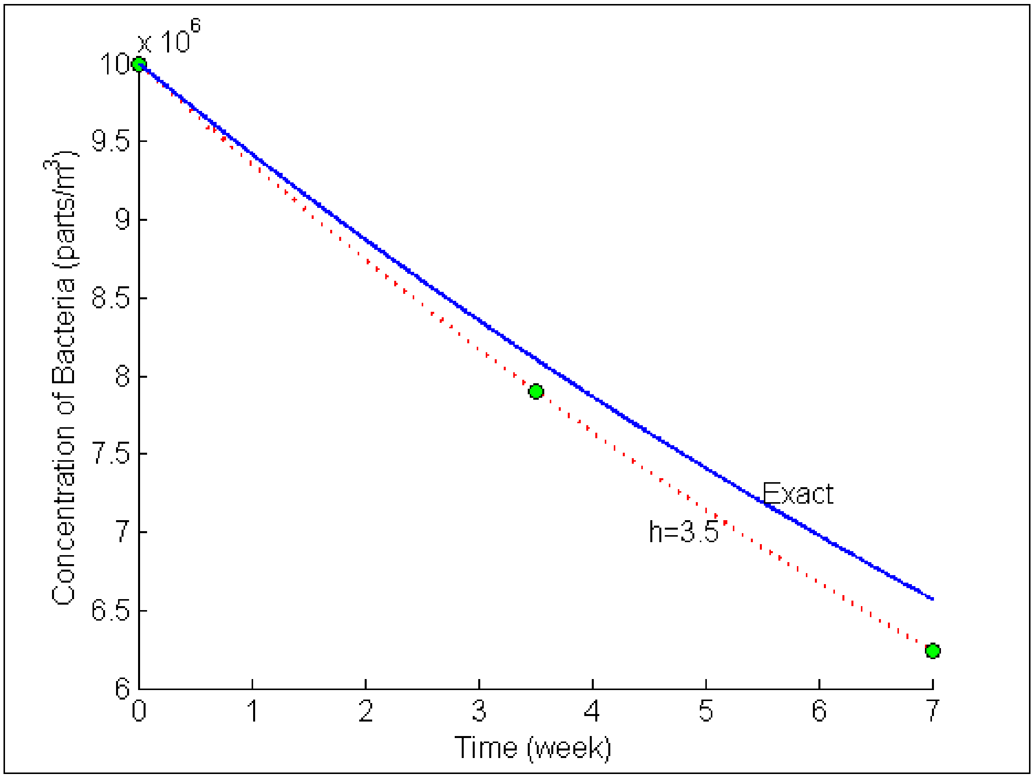

Figure 3 Comparing exact and Euler's method.

The problem was solved again using smaller step sizes. The results are given below in Table 1.

Table 1 Concentration of bacteria after 7 weeks as a function of step size, h.

step size, h	$C(7)$	E_t	$\lvert\in_t\rvert\%$
7	5.8×10^6	770470	11.726
3.5	6.241×10^6	329470	5.0144
1.75	6.4164×10^6	154060	2.3447
0.875	6.4959×10^6	74652	1.1362
0.4375	6.5337×10^6	36763	0.55952

Figure 4 shows how the concentration of bacteria varies as a function of time for different step sizes.

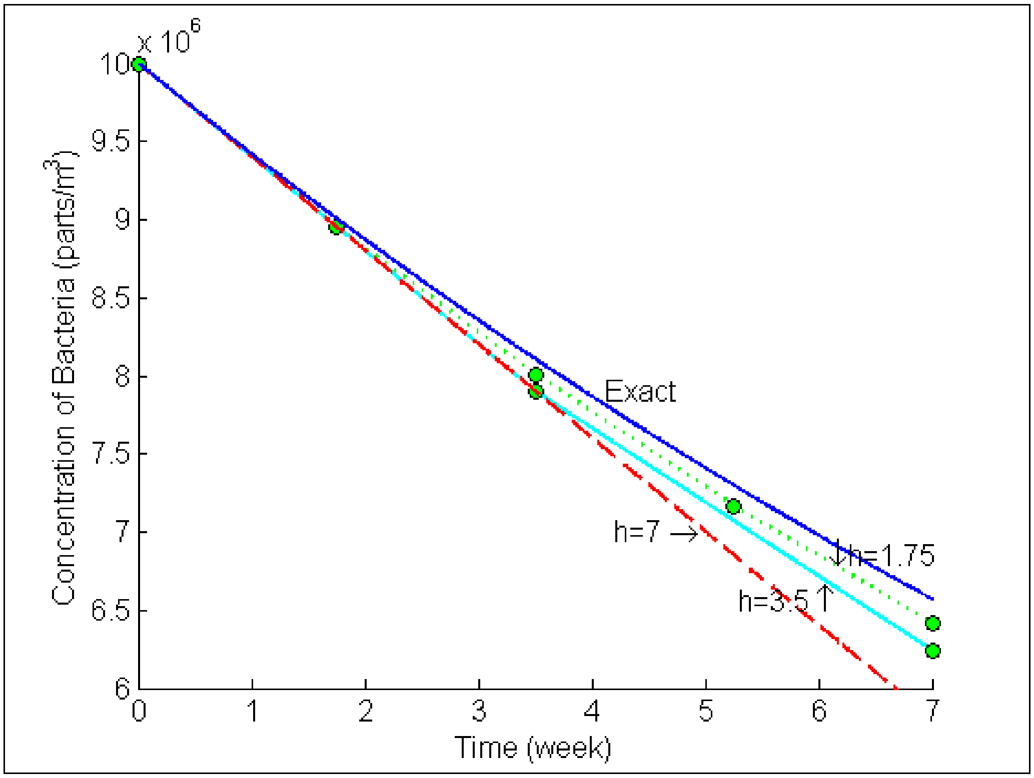

Figure 4 Comparison of Euler's method with exact solution for different step sizes.

While the values of the calculated concentration of bacteria at $t = 7$ weeks as a function of step size are plotted in Figure 5.

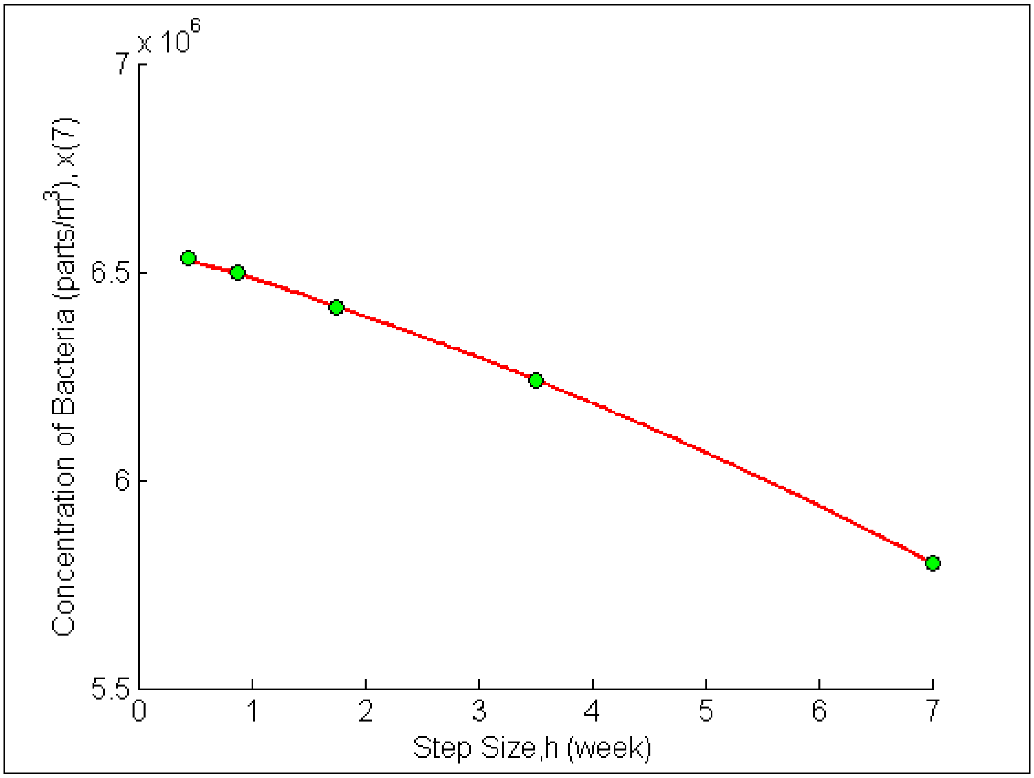

Figure 5 Effect of step size in Euler's method.

The exact solution of the ordinary differential equation is given by

$$C(t) = 1\times 10^{7} e^{\left(\frac{-3t}{50}\right)} \tag{4}$$

The solution to this nonlinear equation at $t = 7$ weeks is

$$C(7) = 6.5705\times 10^{6} \text{ parts/m}^{3}$$

It can be seen that Euler's method has large errors. This can be illustrated using the Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i,y_i}(x_{i+1}-x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i,y_i}(x_{i+1}-x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i,y_i}(x_{i+1}-x_i)^3 + \ldots \tag{5}$$

$$= y_i + f(x_i,y_i)(x_{i+1}-x_i) + \frac{1}{2!}f'(x_i,y_i)(x_{i+1}-x_i)^2 + \frac{1}{3!}f''(x_i,y_i)(x_{i+1}-x_i)^3 + \ldots \tag{6}$$

As you can see the first two terms of the Taylor series

$$y_{i+1} = y_i + f(x_i,y_i)h$$

are Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i,y_i)}{2!}h^2 + \frac{f''(x_i,y_i)}{3!}h^3 + \ldots \tag{7}$$

The true error hence is approximately proportional to the square of the step size, that is, as the step size is halved, the true error gets approximately quartered. However from Table 1, we see that as the step size gets halved, the true error only gets approximately halved. This is because the true error, being proportioned to the square of the step size, is the local truncation error, that is, error from one point to the next. The global truncation error is however proportional only to the step size as the error keeps propagating from one point to another.

Can one solve a definite integral using numerical methods such as Euler's method of solving ordinary differential equations?

Let us suppose you want to find the integral of a function $f(x)$

$$I = \int_a^b f(x)dx .$$

Both fundamental theorems of calculus would be used to set up the problem so as to solve it as an ordinary differential equation.

The first fundamental theorem of calculus states that if f is a continuous function in the interval [a,b], and F is the antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

The second fundamental theorem of calculus states that if f is a continuous function in the open interval D, and a is a point in the interval D, and if

$$F(x) = \int_a^x f(t)dt$$

then

$$F'(x) = f(x)$$

at each point in D.

Asked to find $\int_a^b f(x)dx$, we can rewrite the integral as the solution of an ordinary differential equation (here is where we are using the second fundamental theorem of calculus)

$$\frac{dy}{dx} = f(x), \; y(a) = 0,$$

where then $y(b)$ (here is where we are using the first fundamental theorem of calculus) will give the value of the integral $\int_a^b f(x)dx$.

Example 4

Find an approximate value of

$$\int_5^8 6x^3 dx$$

using Euler's method of solving an ordinary differential equation. Use a step size of $h = 1.5$.

Solution

Given $\int_5^8 6x^3 dx$, we can rewrite the integral as the solution of an ordinary differential equation

$$\frac{dy}{dx} = 6x^3,\ y(5) = 0$$

where $y(8)$ will give the value of the integral $\int_5^8 6x^3 dx$.

$$\frac{dy}{dx} = 6x^3 = f(x, y),\ y(5) = 0$$

The Euler's method equation is

$$y_{i+1} = y_i + f(x_i, y_i)h$$

<u>Step 1</u>

$$i = 0,\ x_0 = 5,\ y_0 = 0$$

$$h = 1.5$$

$$\begin{aligned} x_1 &= x_0 + h \\ &= 5 + 1.5 \\ &= 6.5 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)h \\ &= 0 + f(5,0) \times 1.5 \\ &= 0 + (6 \times 5^3) \times 1.5 \\ &= 1125 \\ &\approx y(6.5) \end{aligned}$$

<u>Step 2</u>

$$i = 1,\ x_1 = 6.5,\ y_1 = 1125$$

$$\begin{aligned} x_2 &= x_1 + h \\ &= 6.5 + 1.5 \\ &= 8 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + f(x_1, y_1)h \\ &= 1125 + f(6.5,1125) \times 1.5 \\ &= 1125 + (6 \times 6.5^3) \times 1.5 \\ &= 3596.625 \\ &\approx y(8) \end{aligned}$$

Hence

$$\begin{aligned} \int_5^8 6x^3 dx &= y(8) - y(5) \\ &\approx 3596.625 - 0 \\ &= 3596.625 \end{aligned}$$

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Euler's Method for ordinary differential equations
Summary	Textbook notes on Euler's method for solving ordinary differential equations
Major	Civil Engineering
Authors	Autar Kaw
Last Revised	September 3, 2009
Web Site	http://numericalmethods.eng.usf.edu