

Chapter 08.04

Runge-Kutta 4th Order Method for Ordinary Differential Equations

After reading this chapter, you should be able to

1. *develop Runge-Kutta 4th order method for solving ordinary differential equations,*
2. *find the effect size of step size has on the solution,*
3. *know the formulas for other versions of the Runge-Kutta 4th order method*

What is the Runge-Kutta 4th order method?

Runge-Kutta 4th order method is a numerical technique used to solve ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

So only first order ordinary differential equations can be solved by using the Runge-Kutta 4th order method. In other sections, we have discussed how Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example 2

Rewrite

$$e^y \frac{dy}{dx} + x^2 y^2 = 2 \sin(3x), y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), y(0) = y_0 \text{ form.}$$

Solution

$$e^y \frac{dy}{dx} + x^2 y^2 = 2 \sin(3x), y(0) = 5$$

$$\frac{dy}{dx} = \frac{2 \sin(3x) - x^2 y^2}{e^y}, y(0) = 5$$

In this case

$$f(x, y) = \frac{2 \sin(3x) - x^2 y^2}{e^y}$$

The Runge-Kutta 4th order method is based on the following

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2 + a_3 k_3 + a_4 k_4)h \quad (1)$$

where knowing the value of $y = y_i$ at x_i , we can find the value of $y = y_{i+1}$ at x_{i+1} , and

$$h = x_{i+1} - x_i$$

Equation (1) is equated to the first five terms of Taylor series

$$y_{i+1} = y_i + \frac{dy}{dx} \Big|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!} \frac{d^2 y}{dx^2} \Big|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!} \frac{d^3 y}{dx^3} \Big|_{x_i, y_i} (x_{i+1} - x_i)^3 + \frac{1}{4!} \frac{d^4 y}{dx^4} \Big|_{x_i, y_i} (x_{i+1} - x_i)^4 \quad (2)$$

Knowing that $\frac{dy}{dx} = f(x, y)$ and $x_{i+1} - x_i = h$

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i)h^2 + \frac{1}{3!} f''(x_i, y_i)h^3 + \frac{1}{4!} f'''(x_i, y_i)h^4 \quad (3)$$

Based on equating Equation (2) and Equation (3), one of the popular solutions used is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \quad (4)$$

$$k_1 = f(x_i, y_i) \quad (5a)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h\right) \quad (5b)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h\right) \quad (5c)$$

$$k_4 = f(x_i + h, y_i + k_3h) \quad (5d)$$

Example 3

A polluted lake has an initial concentration of a bacteria of 10^7 parts/m³, while the acceptable level is only 5×10^6 parts/m³. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration C of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt} + 0.06C = 0, C(0) = 10^7$$

Using the Runge-Kutta 4th order method, find the concentration of the pollutant after 7 weeks. Take a step size of 3.5 weeks.

Solution

$$\frac{dC}{dt} = -0.06C$$

$$f(t, C) = -0.06C$$

$$C_{i+1} = C_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

For $i = 0$, $t_0 = 0$, $C_0 = 10^7$

$$k_1 = f(t_0, C_0)$$

$$= f(0, 10^7)$$

$$= -0.06(10^7)$$

$$= -600000$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, C_0 + \frac{1}{2}k_1h\right)$$

$$= f\left(0 + \frac{1}{2} \times 3.5, 10^7 + \frac{1}{2}(-600000)3.5\right)$$

$$= f(1.75, 8950000)$$

$$= -0.06(8950000)$$

$$= -537000$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, C_0 + \frac{1}{2}k_2h\right)$$

$$= f\left(0 + \frac{1}{2}3.5, 10^7 + \frac{1}{2}(-537000)3.5\right)$$

$$\begin{aligned}
&= f(1.75, 9060300) \\
&= -0.06(9060300) \\
&= -543620 \\
k_4 &= f(t_0 + h, C_0 + k_3 h) \\
&= f(0 + 3.5, 10^7 + (-543620)3.5) \\
&= f(3.5, 8097300) \\
&= -0.06(8097300) \\
&= -485840 \\
C_1 &= C_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\
&= 10^7 + \frac{1}{6}(-600000 + 2(-537000) + 2(-543620) + (-485840))3.5 \\
&= 10^7 + \frac{1}{6}(-3247100)3.5 \\
&= 8.1059 \times 10^6 \text{ parts/m}^3
\end{aligned}$$

C_1 is the approximate concentration of bacteria at

$$t = t_1 = t_0 + h = 0 + 3.5 = 3.5 \text{ parts/m}^3$$

$$C(3.5) \approx C_1 = 8.1059 \times 10^6 \text{ parts/m}^3$$

For $i = 1, t_1 = 3.5, C_1 = 8.1059 \times 10^6$

$$\begin{aligned}
k_1 &= f(t_1, C_1) \\
&= f(3.5, 8.1059 \times 10^6) \\
&= -0.06(8.1059 \times 10^6) \\
&= -486350
\end{aligned}$$

$$\begin{aligned}
k_2 &= f\left(t_1 + \frac{1}{2}h, C_1 + \frac{1}{2}k_1 h\right) \\
&= f\left(3.5 + \frac{1}{2} \times 3.5, 8105900 + \frac{1}{2}(-486350)3.5\right) \\
&= f(5.25, 7254800) \\
&= -0.06(7254800) \\
&= -435290
\end{aligned}$$

$$\begin{aligned}
k_3 &= f\left(t_1 + \frac{1}{2}h, C_1 + \frac{1}{2}k_2 h\right) \\
&= f\left(3.5 + \frac{1}{2} \times 3.5, 8105900 + \frac{1}{2}(-435290)3.5\right) \\
&= f(5.25, 7344100) \\
&= -0.06(7344100) \\
&= -440648
\end{aligned}$$

$$\begin{aligned}
k_4 &= f(t_1 + h, C_1 + k_3 h) \\
&= f(3.5 + 3.5, 8105900 + (-440648)3.5)
\end{aligned}$$

$$\begin{aligned}
 &= f(7, 6563600) \\
 &= -0.06(6563600) \\
 &= -393820 \\
 C_2 &= C_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\
 &= 8105900 + \frac{1}{6}(-486350 + 2 \times (-435290) + 2 \times (-440648) + (-393820)) \times 3.5 \\
 &= 8105900 + \frac{1}{6}(-2632000) \times 3.5 \\
 &= 6.5705 \times 10^6 \text{ parts/m}^3
 \end{aligned}$$

C_2 is the approximate concentration of bacteria at

$$t_2 = t_1 + h = 3.5 + 3.5 = 7 \text{ weeks}$$

$$C(7) \approx C_2 = 6.5705 \times 10^6 \text{ parts/m}^3$$

Figure 1 compares the exact solution with the numerical solution using the Runge-Kutta 4th order method using different step sizes.

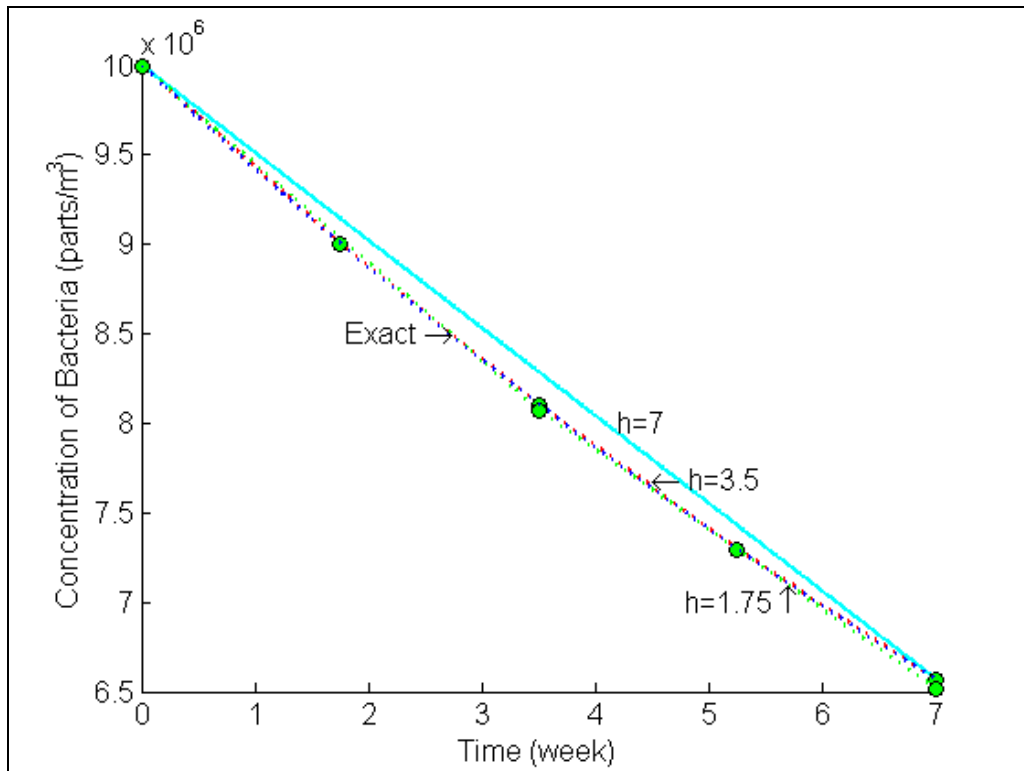
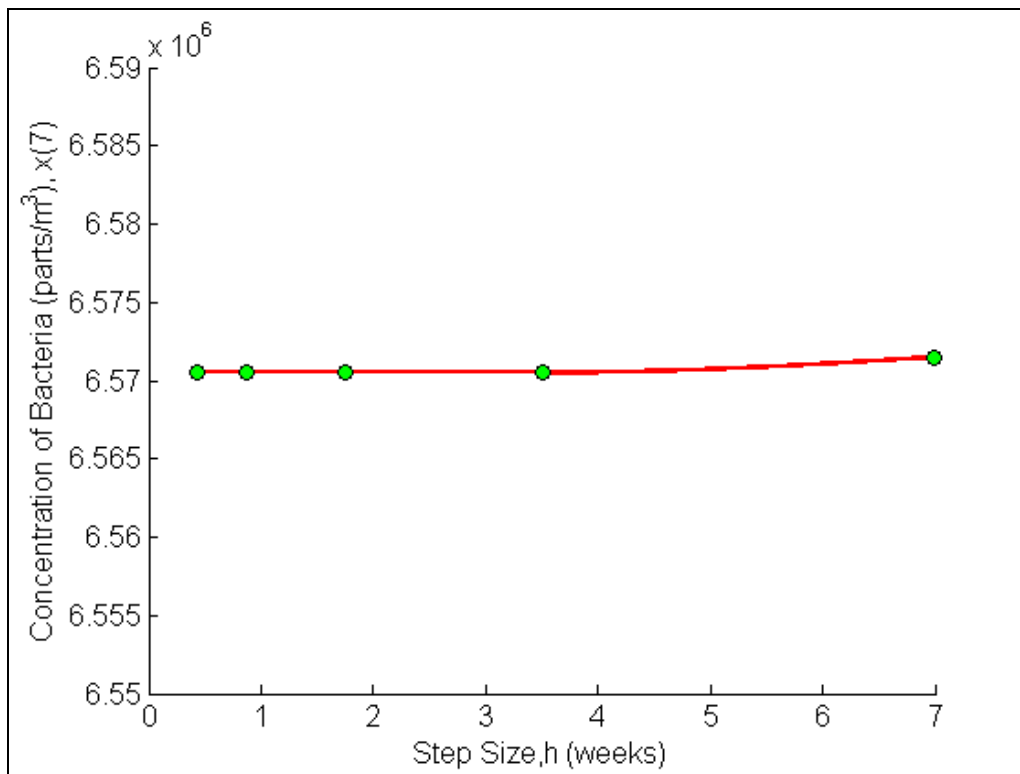


Figure 1 Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.

Table 1 and Figure 2 shows the effect of step size on the value of the calculated concentration of bacteria at $t = 7$ weeks.

Table 1 Value of concentration of bacteria at 7 weeks for different step sizes.

Step size, h	$C(7)$	E_t	$ \epsilon_t \%$
7	6.5715×10^6	-1017.2	0.015481
3.5	6.5705×10^6	-53.301	8.1121×10^{-4}
1.75	6.5705×10^6	-3.0512	4.6438×10^{-5}
0.875	6.5705×10^6	-0.18252	2.7779×10^{-6}
0.4375	6.5705×10^6	-0.011161	1.6986×10^{-7}

**Figure 2** Effect of step size in Runge-Kutta 4th order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method) and the Runge-Kutta 4th order method.

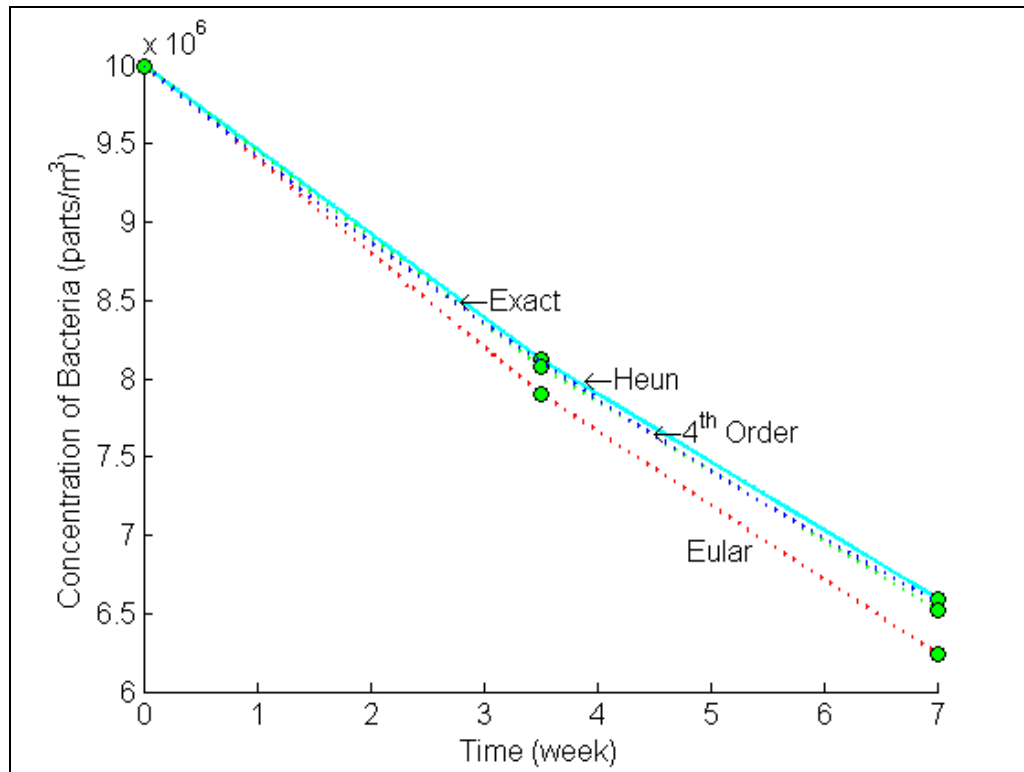


Figure 3 Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method), and Runge-Kutta 4th order method.

The formula described in this chapter was developed by Runge. This formula is same as Simpson's 1/3 rule, if $f(x, y)$ were only a function of x . There are other versions of the 4th order method just like there are several versions of the second order methods. The formula developed by Kutta is

$$y_{i+1} = y_i + \frac{1}{8}(k_1 + 3k_2 + 3k_3 + k_4)h \quad (6)$$

where

$$k_1 = f(x_i, y_i) \quad (7a)$$

$$k_2 = f\left(x_i + \frac{1}{3}h, y_i + \frac{1}{3}hk_1\right) \quad (7b)$$

$$k_3 = f\left(x_i + \frac{2}{3}h, y_i - \frac{1}{3}hk_1 + hk_2\right) \quad (7c)$$

$$k_4 = f(x_i + h, y_i + hk_1 - hk_2 + hk_3) \quad (7d)$$

This formula is the same as the Simpson's 3/8 rule, if $f(x, y)$ is only a function of x .

The exact solution of the ordinary differential equation is given by

$$C(t) = 1 \times 10^7 e^{\left(\frac{-3t}{50}\right)}$$

The solution to this nonlinear equation at $t = 7$ weeks is

$$C(7) = 6.5705 \times 10^6 \text{ parts/m}^3$$

ORDINARY DIFFERENTIAL EQUATIONS

Topic	Runge-Kutta 4th order method
Summary	Textbook notes on the Runge-Kutta 4th order method for solving ordinary differential equations.
Major	Civil Engineering
Authors	Autar Kaw
Last Revised	September 3, 2009
Web Site	http://numericalmethods.eng.usf.edu
