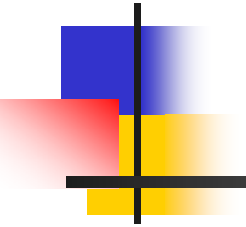




Differentiation-Continuous Functions



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Forward Difference Approximation

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite ' Δx '

$$f'(x) \cong \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Graphical Representation Of Forward Difference Approximation

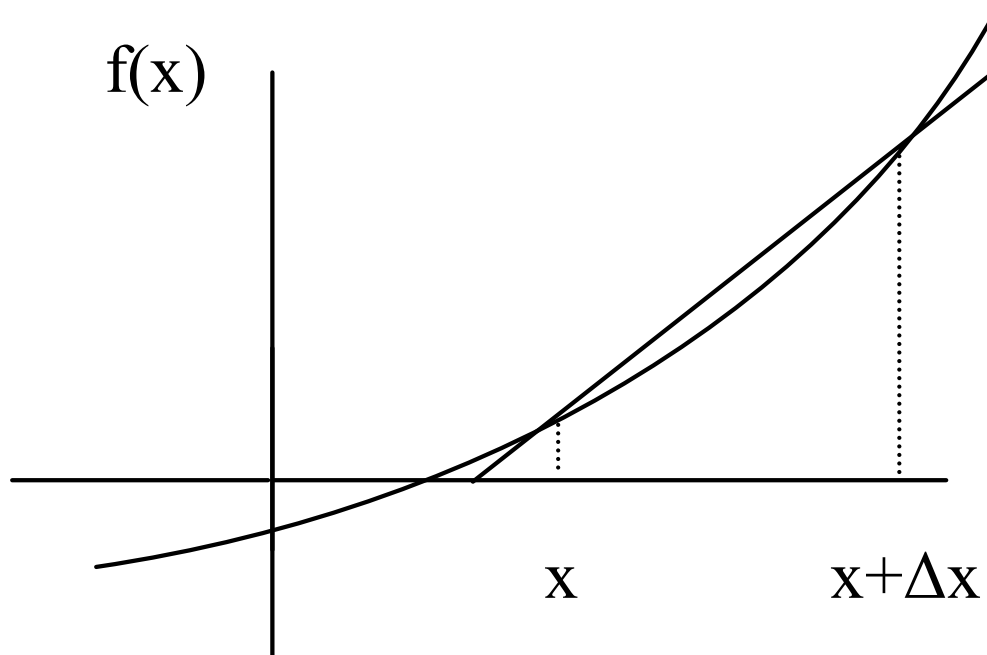


Figure 1: Graphical Representation of forward difference approximation of first derivative



Example 1

There is strong evidence that the first level of processing what we see is done in the retina involves detecting something called *edges* or positions of transitions from dark to bright or bright to dark points in the images. These points usually coincide with boundaries of things. To model the *edges*, derivatives of functions such as

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$$

need to be found.

- a) Use forward divided difference approximation of the first derivative of $f(x)$ to calculate its derivative at for $a=0.24$. Use a step size of $\Delta x = 0.05$
- b) Repeat the procedure for same data except choose $a=0.12$. Does the estimate of the derivative increase or decrease?



Example 1 Cont.

$$f'(x_i) \cong \frac{f(x_{i+1}) - f(x_i)}{\Delta x}$$

$$x_i = 0.1$$

$$\begin{aligned} x_{i+1} &= x + \Delta x = 0.1 + 0.05 \\ &= 0.15 \end{aligned}$$

$$\begin{aligned} f(0.1) &= 1 - e^{(-24 \times 0.1)} \\ &= 0.023714 \end{aligned}$$

$$\begin{aligned} f(0.15) &= 1 - e^{(-24 \times 0.15)} \\ &= 0.035359 \end{aligned}$$



Example 1 Cont.

$$\begin{aligned}f'(0.1) &\approx \frac{f(0.15) - f(0.1)}{0.05} \\ &= \frac{0.035359 - 0.023714}{0.05} \\ &= 0.23291\end{aligned}$$

The exact value of $f'(0.1)$ can be calculated by differentiating

$$f(x) = 1 - e^{ax}, x \geq 0$$

as

$$f'(x) = \frac{d}{dx}[f(x)]$$



Example 1 Cont.

Knowing that

$$\frac{d}{dx} [e^{-ax}] = -ae^{-ax}$$

$$f'(x) = \frac{d}{dx} (1 - e^{-ax})$$

$$= ae^{-ax}$$

$$f'(0.1) = (0.24)(e^{-(0.24 \times 0.1)})$$

$$= 0.23430$$



Example 1 Cont.

The absolute relative true error is

$$|\epsilon_t| = \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100$$

$$= \left| \frac{0.23430 - 0.23291}{0.23430} \right| \times 100$$

$$= 0.59665\%$$



Example 1 Cont.

b)

$$f'(x_i) \cong \frac{f(x_{i+1}) - f(x_i)}{\Delta x}$$

$$x_i = 0.1$$

$$\begin{aligned} x_{i+1} &= x + \Delta x = 0.1 + 0.05 \\ &= 0.15 \end{aligned}$$

$$\begin{aligned} f(0.1) &= 1 - e^{-(0.12 \times 0.1)} \\ &= 0.011928 \end{aligned}$$

$$\begin{aligned} f(0.15) &= 1 - e^{-(0.12 \times 0.15)} \\ &= 0.017838 \end{aligned}$$



Example 1 Cont.

$$\begin{aligned}f'(0.1) &\approx \frac{f(0.15) - f(0.1)}{0.05} \\ &= \frac{0.017838 - 0.011928}{0.05} \\ &= 0.11821\end{aligned}$$

The exact value of $f'(0.1)$ can be calculated by differentiating

$$f(x) = 1 - e^{ax}, x \geq 0$$

as

$$f'(x) = \frac{d}{dx}[f(x)]$$



Example 1 Cont.

Knowing that

$$\frac{d}{dx} [e^{-ax}] = -ae^{-ax}$$

$$f'(x) = \frac{d}{dx} (1 - e^{-ax})$$

$$= ae^{-ax}$$

$$f'(0.1) = (0.12)(e^{-(0.12 \times 0.1)})$$

$$= 0.11856$$



Example 1 Cont.

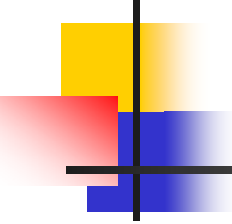
The absolute relative true error is

$$|\epsilon_t| = \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100$$

$$= \left| \frac{0.11856 - 0.11821}{0.11856} \right| \times 100$$

$$= 0.29856\%$$

The estimate of the derivative decreased.



Backward Difference Approximation of the First Derivative

We know

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite ' Δx ',

$$f'(x) \cong \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If ' Δx ' is chosen as a negative number,

$$\begin{aligned} f'(x) &\cong \frac{f(x - \Delta x) - f(x)}{-\Delta x} \\ &= \frac{f(x) - f(x - \Delta x)}{\Delta x} \end{aligned}$$



Backward Difference Approximation of the First Derivative Cont.

This is a backward difference approximation as you are taking a point backward from x . To find the value of $f'(x)$ at $x = x_i$, we may choose another point ' Δx ' behind as $x = x_{i-1}$. This gives

$$\begin{aligned} f'(x_i) &\cong \frac{f(x_i) - f(x_{i-1})}{\Delta x} \\ &= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \end{aligned}$$

where

$$\Delta x = x_i - x_{i-1}$$

Backward Difference Approximation of the First Derivative Cont.

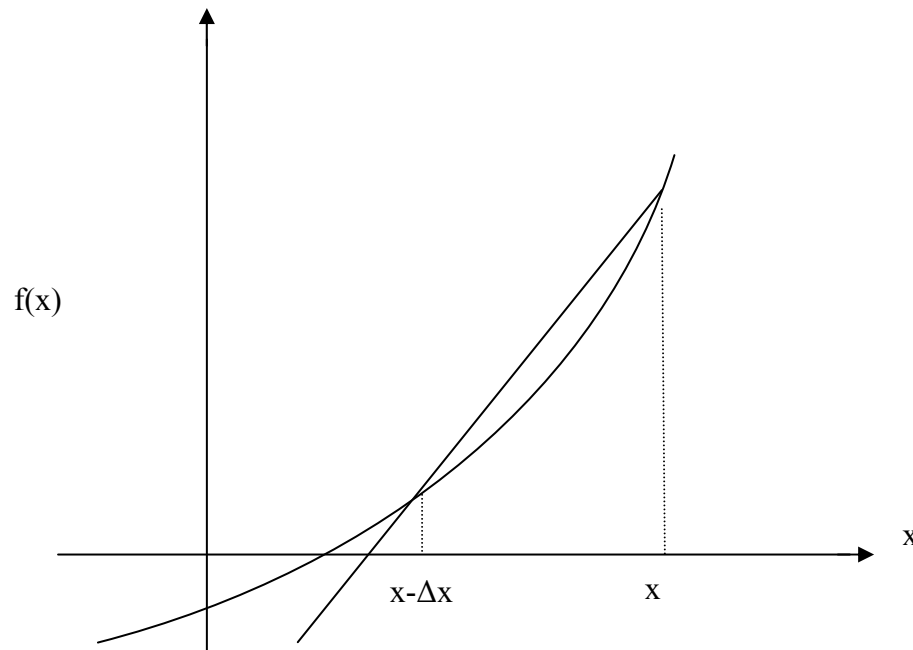


Figure 2: Graphical Representation of backward difference approximation of first derivative



Example 2

There is strong evidence that the first level of processing what we see is done in the retina involves detecting something called *edges* or positions of transitions from dark to bright or bright to dark points in the images. These points usually coincide with boundaries of things. To model the *edges*, derivatives of functions such as

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$$

need to be found.

- a) Use backward divided difference approximation of the first derivative of $f(x)$ to calculate its derivative at $x = 0.1$ for $a=0.24$. Use a step size of $\Delta x = 0.05$



Example 2 Cont.

b) Repeat the procedure for same data except choose $a=0.12$. Does the estimate of the derivative increase or decrease?

Solution

$$a) f'(x_i) \cong \frac{f(x_i) - f(x_{i-1})}{\Delta x}$$

$$x_i = 0.1$$

$$\Delta x = 0.05$$

$$\begin{aligned} x_{i-1} &= 0.1 - 0.05 \\ &= 0.05 \end{aligned}$$

$$f'(0.1) = \frac{f(0.1) - f(0.05)}{0.05}$$



Example 2 Cont.

$$\begin{aligned}f(0.1) &= 1 - e^{-(0.24 \times 0.1)} \\ &= 0.023714\end{aligned}$$

$$\begin{aligned}f(0.05) &= 1 - e^{-(0.24 \times 0.05)} \\ &= 0.011928\end{aligned}$$

$$\begin{aligned}f'(0.1) &= \frac{f(0.1) - f(0.05)}{0.05} \\ &= \frac{0.023714 - 0.011928}{0.05} \\ &= 0.23572\end{aligned}$$



Example 2 Cont.

The absolute relative true error is

$$\begin{aligned} |\epsilon_t| &= \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100 \\ &= \left| \frac{0.234308 - 0.23572}{0.234308} \right| \times 100 \\ &= 0.60262\% \end{aligned}$$



Example 2 Cont.

$$\text{b) } f(x_i) \cong \frac{f(x_i) - f(x_{i-1})}{\Delta x}$$

$$x_i = 0.1$$

$$\Delta x = 0.05$$

$$x_{i-1} = 0.1 - 0.05 = 0.05$$

$$f'(0.1) = \frac{f(0.1) - f(0.05)}{0.05}$$

$$\begin{aligned} f(0.1) &= 1 - e^{-(0.12 \times 0.1)} \\ &= 0.011928 \end{aligned}$$

$$\begin{aligned} f(0.05) &= 1 - e^{-(0.12 \times 0.05)} \\ &= 0.0059820 \end{aligned}$$



Example 2 Cont.

$$\begin{aligned}f'(0.1) &= \frac{f(0.1) - f(0.05)}{0.05} \\ &= \frac{0.0119282 - 0.00598203}{0.05} \\ &= 0.11892\end{aligned}$$

The absolute relative true error is

$$\begin{aligned}|\epsilon_t| &= \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100 \\ &= \left| \frac{0.11856 - 0.11892}{0.11856} \right| \times 100 \\ &= 0.29940\%\end{aligned}$$

The estimate of the derivative decreased.



Derive the forward difference approximation from Taylor series

Taylor's theorem says that if you know the value of a function f at a point x_i and all its derivatives at that point, provided the derivatives are continuous between x_i and x_{i+1} , then

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \dots$$

Substituting for convenience $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} - \frac{f''(x_i)}{2!}(\Delta x) + \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x)$$



Derive the forward difference approximation from Taylor series Cont.

The $(O\Delta x)$ term shows that the error in the approximation is of the order of (Δx) . Can you now derive from Taylor series the formula for backward divided difference approximation of the first derivative?

As shown above, both forward and backward divided difference approximation of the first derivative are accurate on the order of $(O\Delta x)$.

Can we get better approximations? Yes, another method to approximate the first derivative is called the **Central difference approximation of the first derivative**.



From Taylor series

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f'''(x_i)}{3!}(\Delta x)^3 + \dots$$

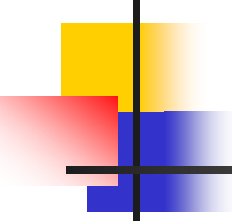
$$f(x_{i-1}) = f(x_i) - f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f'''(x_i)}{3!}(\Delta x)^3 + \dots$$

Subtracting equation (2) from equation (1)

$$f(x_{i+1}) - f(x_{i-1}) = f'(x_i)(2\Delta x) + \frac{2f'''(x_i)}{3!}(\Delta x)^3 + \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} - \frac{f'''(x_i)}{3!}(\Delta x)^2 + \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O(\Delta x)^2$$



hence showing that we have obtained a more accurate formula as the error is of the order of $O(\Delta x)^2$

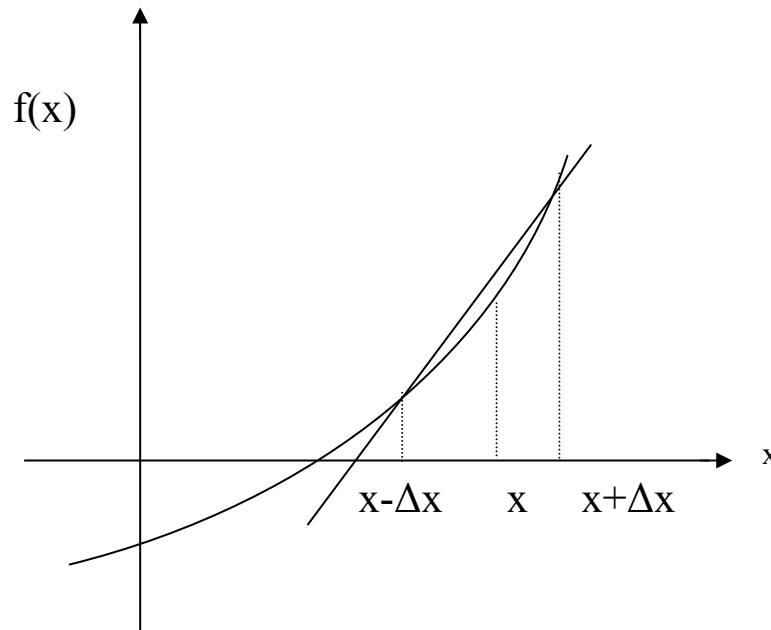


Figure 3: Graphical Representation of central difference approximation of first derivative



Example 3

There is strong evidence that the first level of processing what we see is done in the retina involves detecting something called *edges* or positions of transitions from dark to bright or bright to dark points in the images. These points usually coincide with boundaries of things. To model the *edges*, derivatives of functions such as

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$$

need to be found.

- a) Use central divided difference approximation of the first derivative of $f(x)$ to calculate its derivative at $x = 0.1$ for $a=0.24$. Use a step size of $\Delta x = 0.05$



Example 3 cont.

b) Repeat the procedure for same data except choose $a=0.12$. Does the estimate of the derivative increase or decrease?

Solution

$$\text{a) } f'(x_i) \cong \frac{f(t_{x+1}) - f(t_{x-1})}{2\Delta x}$$

$$x_i = 0.1$$

$$x_{i+1} = x_i + \Delta x$$

$$= 0.1 + 0.05$$

$$= 0.15$$

$$x_{i-1} = x_i - \Delta x$$

$$= 0.1 - 0.05$$

$$= 0.05$$



Example 3 cont.

$$\begin{aligned}f'(0.1) &= \frac{f(0.15) - v(0.05)}{2(0.05)} \\ &= \frac{f(0.15) - f(0.05)}{0.1}\end{aligned}$$

$$\begin{aligned}f(0.15) &= 1 - e^{-(0.24 \times 0.15)} \\ &= 0.035359\end{aligned}$$

$$\begin{aligned}f(0.05) &= 1 - e^{-(0.24 \times 0.05)} \\ &= 0.011928\end{aligned}$$



Example 3 cont.

$$\begin{aligned}f'(0.1) &= \frac{f(0.15) - f(0.05)}{0.1} \\ &= \frac{0.035359 - 0.011928}{0.1} \\ &= 0.23431\end{aligned}$$

The absolute relative true error is

$$\begin{aligned}|\epsilon_t| &= \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100 \\ &= \left| \frac{0.234308 - 0.234315}{0.234308} \right| \times 100 \\ &= 0.0029875\%\end{aligned}$$



Example 3 cont.

$$f'(x_i) \cong \frac{f(t_{x+1}) - f(t_{x-1})}{2\Delta x}$$

$$x_i = 0.1$$

$$\begin{aligned}x_{i+1} &= x_i + \Delta x \\ &= 0.1 + 0.05 \\ &= 0.15\end{aligned}$$

$$\begin{aligned}x_{i-1} &= x_i - \Delta x \\ &= 0.1 - 0.05 \\ &= 0.05\end{aligned}$$



Example 3 cont.

$$\begin{aligned} f'(0.1) &= \frac{f(0.15) - v(0.05)}{2(0.05)} \\ &= \frac{f(0.15) - f(0.05)}{0.1} \end{aligned}$$

$$\begin{aligned} f(0.15) &= 1 - e^{-(0.12 \times 0.15)} \\ &= 0.017838 \end{aligned}$$

$$\begin{aligned} f(0.05) &= 1 - e^{-(0.12 \times 0.05)} \\ &= 0.0059820 \end{aligned}$$



Example 3 cont.

$$\begin{aligned}f'(0.1) &= \frac{f(0.15) - f(0.05)}{0.1} \\ &= \frac{0.017838 - 0.0059820}{0.1} \\ &= 0.11856\end{aligned}$$

The absolute relative true error is

$$\begin{aligned}|\epsilon_t| &= \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100 \\ &= \left| \frac{0.11856 - 0.11856}{0.11856} \right| \times 100 \\ &= 0\%\end{aligned}$$

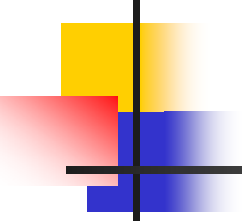


Comparision

The results from the three difference approximations are given in Table 1.

Type of Difference Approximation	$f'(0.1), a = 0.24$	$ \epsilon_t \%, a = 0.24$	$f'(0.1), a = 0.12$	$ \epsilon_t \%, a = 0.12$
Forward	0.23291	0.59665	0.11821	0.29856
Backward	0.23572	0.60262	0.11892	0.29940
Central	0.23431	0.0029875	0.11856	0

Table 1: Summary of $f'(0.1)$ using different divided difference approximations



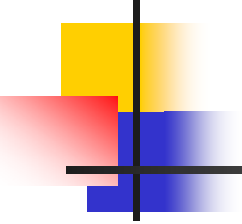
Clearly, the central difference scheme is giving more accurate results because the order of accuracy is proportional to the square of the step size.

In real life, one would not know the exact value of the derivative – so how would one know how accurately they have found the value of the derivative.

A simple way would be to start with a step size and keep on halving the step size and keep on halving the step size until the absolute relative approximate error is within a pre-specified tolerance.

Take the example of finding $f'(x)$ for

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$$



at $x = 0.1$ using the backward divided difference scheme. Given in Table 2 are the values obtained using the backward difference approximation method and the corresponding absolute relative approximate errors.

Δx	$f'(x)$	$ \epsilon_t \%$
0.05	0.23572	0.29875
0.025	0.23500	0.11437
0.0125	0.23457	0.052921
0.00625	0.23443	0.0017071
0.003125	0.23430	