

Chapter 02.02

Differentiation of Continuous Functions

After reading this chapter, you should be able to:

1. *derive formulas for approximating the first derivative of a function,*
2. *derive formulas for approximating derivatives from Taylor series,*
3. *derive finite difference approximations for higher order derivatives, and*
4. *use the developed formulas in examples to find derivatives of a function.*

The derivative of a function at x is defined as

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

To be able to find a derivative numerically, one could make Δx finite to give,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Knowing the value of x at which you want to find the derivative of $f(x)$, we choose a value of Δx to find the value of $f'(x)$. To estimate the value of $f'(x)$, three such approximations are suggested as follows.

Forward Difference Approximation of the First Derivative

From differential calculus, we know

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite Δx ,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The above is the forward divided difference approximation of the first derivative. It is called forward because you are taking a point ahead of x . To find the value of $f'(x)$ at $x = x_i$, we may choose another point Δx ahead as $x = x_{i+1}$. This gives

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{\Delta x}$$

$$= \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

where

$$\Delta x = x_{i+1} - x_i$$

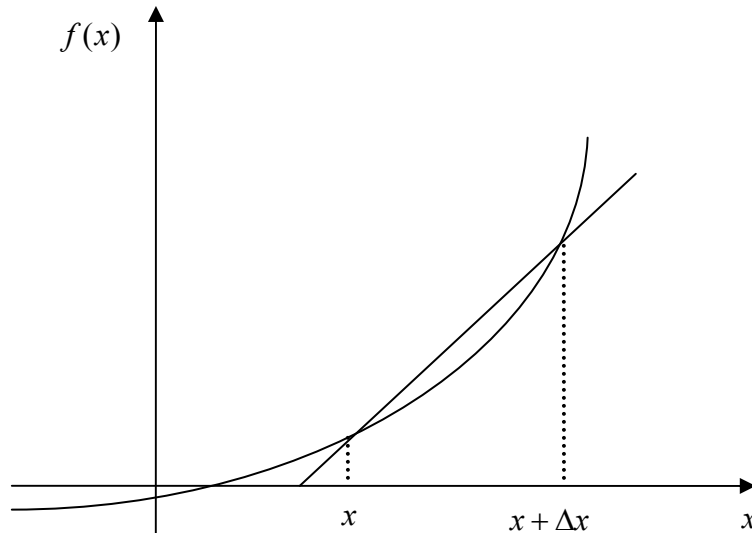


Figure 1 Graphical representation of forward difference approximation of first derivative.

Example 1

There is strong evidence that the first level of processing what we see is done in the retina. It involves detecting something called edges or positions of transitions from dark to bright or bright to dark points in images. These points usually coincide with boundaries of objects. To model the edges, derivatives of functions such as

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$$

need to be found.

- Use the forward divided difference approximation of the first derivative of $f(x)$ to calculate the functions derivative at $x = 0.1$ for $a = 0.24$. Use a step size of $\Delta x = 0.05$. Also, calculate the absolute relative true error.
- Repeat the procedure from part (a) with the same data except choose $a = 0.12$. Does the estimate of the derivative increase or decrease? Also, calculate the absolute relative true error.

Solution

$$\text{a) } f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{\Delta x}$$

$$a = 0.24$$

$$x_i = 0.1$$

$$\Delta x = 0.05$$

$$x_{i+1} = x + \Delta x$$

$$= 0.1 + 0.05$$

$$= 0.15$$

$$f(0.1) = 1 - e^{(-0.24 \times 0.1)}$$

$$= 0.023714$$

$$f(0.15) = 1 - e^{(-0.24 \times 0.15)}$$

$$= 0.035360$$

$$\begin{aligned} f'(0.1) &\approx \frac{f(0.15) - f(0.1)}{0.05} \\ &= \frac{0.035360 - 0.023714}{0.05} \\ &= 0.23291 \end{aligned}$$

The exact value of $f'(0.1)$ can be calculated by differentiating

$$f(x) = 1 - e^{-ax}, \quad x \geq 0$$

as

$$f'(x) = \frac{d}{dx}[f(x)]$$

Knowing that

$$\frac{d}{dx}[e^{-ax}] = -ae^{-ax}$$

we find

$$\begin{aligned} f'(x) &= \frac{d}{dx}(1 - e^{-ax}) \\ &= ae^{-ax} \\ &= 0.24e^{-0.24x} \\ f'(0.1) &= (0.24)(e^{-(0.24 \times 0.1)}) \\ &= 0.23431 \end{aligned}$$

The absolute relative true error is

$$\begin{aligned} |\epsilon_t| &= \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100 \\ &= \left| \frac{0.23431 - 0.23291}{0.23431} \right| \times 100 \\ &= 0.59761\% \end{aligned}$$

b) $a = 0.12$

$$f(0.1) = 1 - e^{-(0.12 \times 0.1)}$$

$$= 0.011928$$

$$\begin{aligned}
 f(0.15) &= 1 - e^{-(0.12 \times 0.15)} \\
 &= 0.017839 \\
 f'(0.1) &\approx \frac{f(0.15) - f(0.1)}{0.05} \\
 &= \frac{0.017839 - 0.011928}{0.05} \\
 &= 0.11821 \\
 f'(x) &= \frac{d}{dx}(1 - e^{-ax}) \\
 &= ae^{-ax} \\
 &= 0.12e^{-0.12x} \\
 f'(0.1) &= (0.12)(e^{-(0.12 \times 0.1)}) \\
 &= 0.11856
 \end{aligned}$$

The absolute relative true error is

$$\begin{aligned}
 |\epsilon_t| &= \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100 \\
 &= \left| \frac{0.11857 - 0.11821}{0.11857} \right| \times 100 \\
 &= 0.29940\%
 \end{aligned}$$

The estimate of the derivative decreased.

Backward Difference Approximation of the First Derivative

We know

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite Δx ,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If Δx is chosen as a negative number,

$$\begin{aligned}
 f'(x) &\approx \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \frac{f(x) - f(x - \Delta x)}{\Delta x}
 \end{aligned}$$

This is a backward difference approximation as you are taking a point backward from x . To find the value of $f'(x)$ at $x = x_i$, we may choose another point Δx behind as $x = x_{i-1}$. This gives

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{\Delta x}$$

$$= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

where

$$\Delta x = x_i - x_{i-1}$$

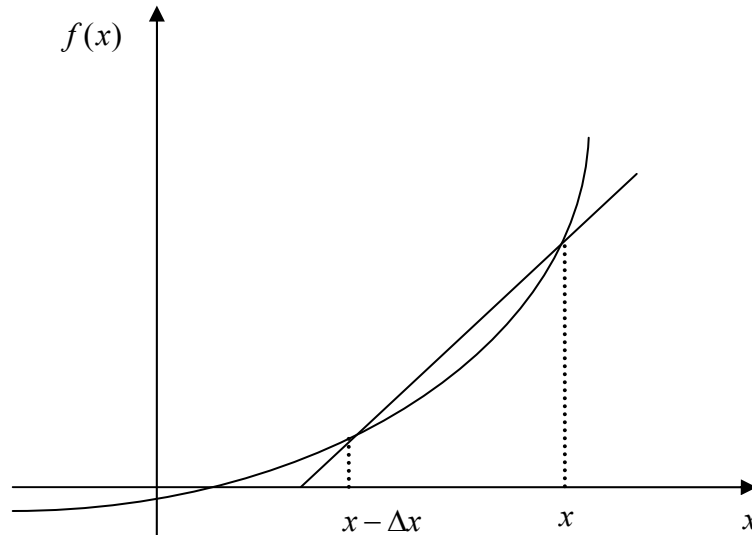


Figure 2 Graphical representation of backward difference approximation of first derivative.

Example 2

There is strong evidence that the first level of processing what we see is done in the retina. It involves detecting something called edges or positions of transitions from dark to bright or bright to dark points in images. These points usually coincide with boundaries of objects. To model the edges, derivatives of functions such as

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$$

need to be found.

- Use the backward divided difference approximation of the first derivative of $f(x)$ to calculate the functions derivative at $x = 0.1$ for $a = 0.24$. Use a step size of $\Delta x = 0.05$. Also, calculate the absolute relative true error.
- Repeat the procedure from part (a) with the same data except choose $a = 0.12$. Does the estimate of the derivative increase or decrease? Also, calculate the absolute relative true error.

Solution

$$a) \quad f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{\Delta x}$$

$$a = 0.24$$

$$x_i = 0.1$$

$$\Delta x = 0.05$$

$$x_{i-1} = x_i - \Delta x$$

$$= 0.1 - 0.05$$

$$= 0.05$$

$$f(0.1) = 1 - e^{-(0.24 \times 0.1)}$$

$$= 0.023714$$

$$f(0.05) = 1 - e^{-(0.24 \times 0.05)}$$

$$= 0.011928$$

$$f'(0.1) \approx \frac{f(0.1) - f(0.05)}{0.05}$$

$$= \frac{0.023714 - 0.011928}{0.05}$$

$$= 0.23572$$

The absolute relative true error is

$$|\epsilon_t| = \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100$$

$$= \left| \frac{0.23431 - 0.23572}{0.23431} \right| \times 100$$

$$= 0.60241\%$$

b) $a = 0.12$

$$f(0.1) = 1 - e^{-(0.12 \times 0.1)}$$

$$= 0.011928$$

$$f(0.05) = 1 - e^{-(0.12 \times 0.05)}$$

$$= 0.0059820$$

$$f'(0.1) \approx \frac{f(0.1) - f(0.05)}{0.05}$$

$$= \frac{0.011928 - 0.0059820}{0.05}$$

$$= 0.11893$$

The absolute relative true error is

$$|\epsilon_t| = \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100$$

$$= \left| \frac{0.11857 - 0.11893}{0.11857} \right| \times 100$$

$$= 0.30060\%$$

The estimate of the derivative decreased.

Forward Difference Approximation from Taylor Series

Taylor's theorem says that if you know the value of a function $f(x)$ at a point x_i and all its derivatives at that point, provided the derivatives are continuous between x_i and x_{i+1} , then

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \dots$$

Substituting for convenience $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} - \frac{f''(x_i)}{2!}(\Delta x) + \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x)$$

The $O(\Delta x)$ term shows that the error in the approximation is of the order of Δx .

Can you now derive from the Taylor series the formula for the backward divided difference approximation of the first derivative?

As you can see, both forward and backward divided difference approximations of the first derivative are accurate on the order of $O(\Delta x)$. Can we get better approximations? Yes, another method to approximate the first derivative is called the **central difference approximation of the first derivative**.

From the Taylor series

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f'''(x_i)}{3!}(\Delta x)^3 + \dots \quad (1)$$

and

$$f(x_{i-1}) = f(x_i) - f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f'''(x_i)}{3!}(\Delta x)^3 + \dots \quad (2)$$

Subtracting Equation (2) from Equation (1)

$$f(x_{i+1}) - f(x_{i-1}) = f'(x_i)(2\Delta x) + \frac{2f'''(x_i)}{3!}(\Delta x)^3 + \dots$$

$$\begin{aligned} f'(x_i) &= \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} - \frac{f'''(x_i)}{3!}(\Delta x)^2 + \dots \\ &= \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O(\Delta x)^2 \end{aligned}$$

hence showing that we have obtained a more accurate formula as the error is of the order of $O(\Delta x)^2$.

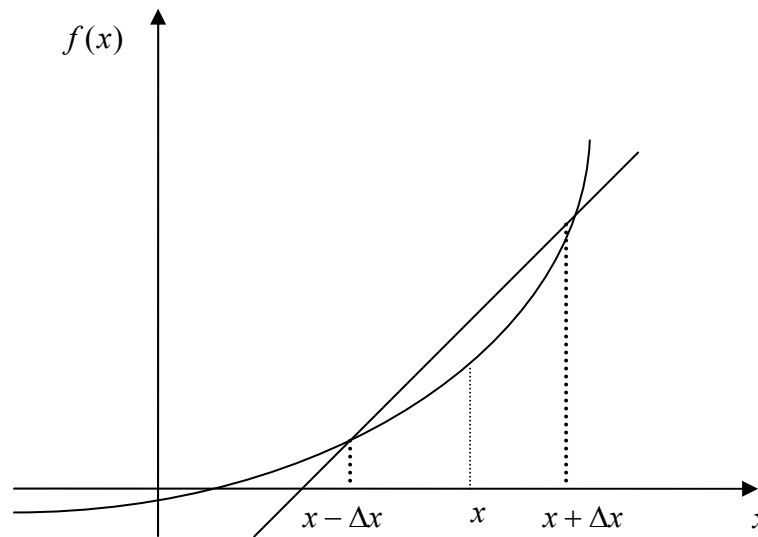


Figure 3 Graphical representation of central difference approximation of first derivative.

Example 3

There is strong evidence that the first level of processing what we see is done in the retina. It involves detecting something called edges or positions of transitions from dark to bright or bright to dark points in images. These points usually coincide with boundaries of objects. To model the edges, derivatives of functions such as

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$$

need to be found.

- Use the central divided difference approximation of the first derivative of $f(x)$ to calculate the functions derivative at $x = 0.1$ for $a = 0.24$. Use a step size of $\Delta x = 0.05$. Also, calculate the absolute relative true error.
- Repeat the procedure from part (a) with the same data except choose $a = 0.12$. Does the estimate of the derivative increase or decrease? Also, calculate the absolute relative true error.

Solution

$$a) \quad f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1}))}{2\Delta x}$$

$$a = 0.24$$

$$x_i = 0.1$$

$$\Delta x = 0.05$$

$$x_{i+1} = x_i + \Delta x$$

$$= 0.1 + 0.05$$

$$= 0.15$$

$$\begin{aligned}
 x_{i-1} &= x_i - \Delta x \\
 &= 0.1 - 0.05 \\
 &= 0.05 \\
 f'(0.1) &\approx \frac{f(0.15) - f(0.05)}{2(0.05)} \\
 &= \frac{f(0.15) - f(0.05)}{0.1}
 \end{aligned}$$

$$\begin{aligned}
 f(0.15) &= 1 - e^{-(0.24 \times 0.15)} \\
 &= 0.035360 \\
 f(0.05) &= 1 - e^{-(0.24 \times 0.05)} \\
 &= 0.011928
 \end{aligned}$$

$$\begin{aligned}
 f'(0.1) &\approx \frac{f(0.15) - f(0.05)}{0.1} \\
 &= \frac{0.035360 - 0.011928}{0.1} \\
 &= 0.23431
 \end{aligned}$$

The absolute relative true error is

$$\begin{aligned}
 |\epsilon_t| &= \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100 \\
 &= \left| \frac{0.23431 - 0.23431}{0.23431} \right| \times 100 \\
 &= 0.0024000\%
 \end{aligned}$$

b) $a = 0.12$

$$\begin{aligned}
 f(0.15) &= 1 - e^{-(0.12 \times 0.15)} \\
 &= 0.017839 \\
 f(0.05) &= 1 - e^{-(0.12 \times 0.05)} \\
 &= 0.0059820
 \end{aligned}$$

$$\begin{aligned}
 f'(0.1) &\approx \frac{f(0.15) - f(0.05)}{0.1} \\
 &= \frac{0.017839 - 0.0059820}{0.1} \\
 &= 0.11857
 \end{aligned}$$

The absolute relative true error is

$$\begin{aligned}
 |\epsilon_t| &= \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100 \\
 &= \left| \frac{0.11857 - 0.11857}{0.11857} \right| \times 100
 \end{aligned}$$

$$= 6.0000 \times 10^{-4}\%$$

The results from the three difference approximations in Examples 1, 2 and 3 are given in Table 1.

Table 1 Summary of $f'(0.1)$ values using different divided difference approximations.

Type of Difference Approximation	$f'(0.1), a = 0.24$	$ \epsilon_i \%, a = 0.24$	$f'(0.1), a = 0.12$	$ \epsilon_i \%, a = 0.12$
Forward	0.23291	0.59761	0.11821	0.29940
Backward	0.23572	0.60241	0.11893	0.30060
Central	0.23431	0.0024000	0.11857	6.0000×10^{-4}

Clearly, the central difference scheme is giving more accurate results because the order of accuracy is proportional to the square of the step size.

In real life, one would not know the exact value of the derivative, so how would one know how accurately they have found the value of the derivative?

A simple way would be to start with a step size and keep on halving the step size until the absolute relative approximate error is within a pre-specified tolerance.

Take the example of finding $f'(x)$ for $f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$ at $x = 0.1$ for $a = 0.24$

using the backward divided difference scheme. Given in Table 2 are the values obtained using the backward divided difference approximation method and the corresponding absolute relative approximate errors.

Table 2 Values of $f'(0.1)$ using backward different divided difference approximation with different step sizes.

Δx	$f'(x)$	$ \epsilon_a \%$
0.05	0.23572	
0.025	0.23501	0.30090
0.0125	0.23466	0.15023
0.00625	0.23448	0.075056
0.003125	0.23440	0.037514

Finite Difference Approximation of Higher Derivatives

One can also use the Taylor series to approximate a higher order derivative. For example, to approximate $f''(x)$, the Taylor series is

$$f(x_{i+2}) = f(x_i) + f'(x_i)(2\Delta x) + \frac{f''(x_i)}{2!}(2\Delta x)^2 + \frac{f'''(x_i)}{3!}(2\Delta x)^3 + \dots \quad (3)$$

where

$$x_{i+2} = x_i + 2\Delta x$$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f'''(x_i)}{3!}(\Delta x)^3 \dots \quad (4)$$

where

$$x_{i-1} = x_i - \Delta x$$

Subtracting 2 times Equation (4) from Equation (3) gives

$$f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + f''(x_i)(\Delta x)^2 + f'''(x_i)(\Delta x)^3 \dots$$

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{(\Delta x)^2} - f'''(x_i)(\Delta x) + \dots$$

$$f''(x_i) \approx \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{(\Delta x)^2} + O(\Delta x) \quad (5)$$

Example 4

The velocity of a rocket is given by

$$v(t) = 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right] - 9.8t, 0 \leq t \leq 30$$

Use the forward difference approximation of the second derivative of $v(t)$ to calculate the jerk at $t = 16$ s. Use a step size of $\Delta t = 2$ s.

Solution

$$j(t_i) \approx \frac{v(t_{i+2}) - 2v(t_{i+1}) + v(t_i)}{(\Delta t)^2}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$t_{i+1} = t_i + \Delta t$$

$$= 16 + 2$$

$$= 18$$

$$t_{i+2} = t_i + 2(\Delta t)$$

$$= 16 + 2(2)$$

$$= 20$$

$$j(16) \approx \frac{v(20) - 2v(18) + v(16)}{(2)^2}$$

$$v(20) = 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100(20)} \right] - 9.8(20)$$

$$= 517.35 \text{ m/s}$$

$$v(18) = 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100(18)} \right] - 9.8(18)$$

$$= 453.02 \text{ m/s}$$

$$v(16) = 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100(16)} \right] - 9.8(16)$$

$$= 392.07 \text{ m/s}$$

$$j(16) \approx \frac{517.35 - 2(453.02) + 392.07}{4}$$

$$= 0.84515 \text{ m/s}^3$$

The exact value of $j(16)$ can be calculated by differentiating

$$v(t) = 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right] - 9.8t$$

twice as

$$a(t) = \frac{d}{dt} [v(t)] \text{ and}$$

$$j(t) = \frac{d}{dt} [a(t)]$$

Knowing that

$$\frac{d}{dt} [\ln(t)] = \frac{1}{t} \text{ and}$$

$$\frac{d}{dt} \left[\frac{1}{t} \right] = -\frac{1}{t^2}$$

$$a(t) = 2000 \left(\frac{14 \times 10^4 - 2100t}{14 \times 10^4} \right) \frac{d}{dt} \left(\frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right) - 9.8$$

$$= 2000 \left(\frac{14 \times 10^4 - 2100t}{14 \times 10^4} \right) (-1) \left(\frac{14 \times 10^4}{(14 \times 10^4 - 2100t)^2} \right) (-2100) - 9.8$$

$$= \frac{-4040 - 29.4t}{-200 + 3t}$$

Similarly it can be shown that

$$j(t) = \frac{d}{dt} [a(t)]$$

$$= \frac{18000}{(-200 + 3t)^2}$$

$$j(16) = \frac{18000}{[-200 + 3(16)]^2}$$

$$= 0.77909 \text{ m/s}^3$$

The absolute relative true error is

$$|\epsilon_i| = \left| \frac{0.77909 - 0.84515}{0.77909} \right| \times 100$$

$$= 8.4797\%$$

The formula given by Equation (5) is a forward difference approximation of the second derivative and has an error of the order of $O(\Delta x)$. Can we get a formula that has a better accuracy? Yes, we can derive the central difference approximation of the second derivative.

The Taylor series is

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f'''(x_i)}{3!}(\Delta x)^3 + \frac{f^{(4)}(x_i)}{4!}(\Delta x)^4 + \dots \quad (6)$$

where

$$x_{i+1} = x_i + \Delta x$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f'''(x_i)}{3!}(\Delta x)^3 + \frac{f^{(4)}(x_i)}{4!}(\Delta x)^4 - \dots \quad (7)$$

where

$$x_{i-1} = x_i - \Delta x$$

Adding Equations (6) and (7), gives

$$f(x_{i+1}) + f(x_{i-1}) = 2f(x_i) + f''(x_i)(\Delta x)^2 + f^{(4)}(x_i)\frac{(\Delta x)^4}{12} + \dots$$

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1}))}{(\Delta x)^2} - \frac{f^{(4)}(x_i)(\Delta x)^2}{12} + \dots$$

$$= \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1}))}{(\Delta x)^2} + O(\Delta x)^2$$

Example 5

The velocity of a rocket is given by

$$v(t) = 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right] - 9.8t, \quad 0 \leq t \leq 30,$$

(a) Use the central difference approximation of the second derivative of $v(t)$ to calculate the jerk at $t = 16$ s. Use a step size of $\Delta t = 2$ s.

Solution

$$a(t_i) \approx \frac{v(t_{i+1}) - 2v(t_i) + v(t_{i-1}))}{(\Delta t)^2}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$t_{i+1} = t_i + \Delta t$$

$$= 16 + 2$$

$$= 18$$

$$\begin{aligned}
 t_{i+2} &= t_i - \Delta t \\
 &= 16 - 2 \\
 &= 14
 \end{aligned}$$

$$j(16) \approx \frac{v(18) - 2v(16) + v(14)}{(2)^2}$$

$$\begin{aligned}
 v(18) &= 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100(18)} \right] - 9.8(18) \\
 &= 453.02 \text{ m/s}
 \end{aligned}$$

$$\begin{aligned}
 v(16) &= 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100(16)} \right] - 9.8(16) \\
 &= 392.07 \text{ m/s}
 \end{aligned}$$

$$\begin{aligned}
 v(14) &= 2000 \ln \left[\frac{14 \times 10^4}{14 \times 10^4 - 2100(14)} \right] - 9.8(14) \\
 &= 334.24 \text{ m/s}
 \end{aligned}$$

$$\begin{aligned}
 j(16) &\approx \frac{v(18) - 2v(16) + v(14)}{(2)^2} \\
 &= \frac{453.02 - 2(392.07) + 334.24}{4} \\
 &= 0.77969 \text{ m/s}^3
 \end{aligned}$$

The absolute relative true error is

$$\begin{aligned}
 |\epsilon_t| &= \left| \frac{0.77908 - 0.77969}{0.77908} \right| \times 100 \\
 &= 0.077992\%
 \end{aligned}$$

DIFFERENTIATION

Topic	Differentiation of Continuous functions
Summary	These are textbook notes of differentiation of continuous functions
Major	Computer Engineering
Authors	Autar Kaw, Luke Snyder
Date	June 20, 2009
Web Site	http://numericalmethods.eng.usf.edu
