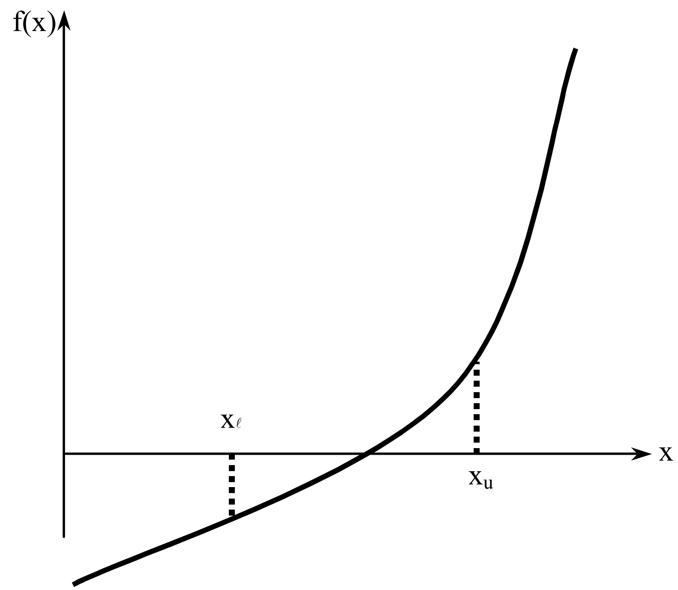
Roots of a Nonlinear Equation

Topic: Bisection Method

Major: Computer Engineering

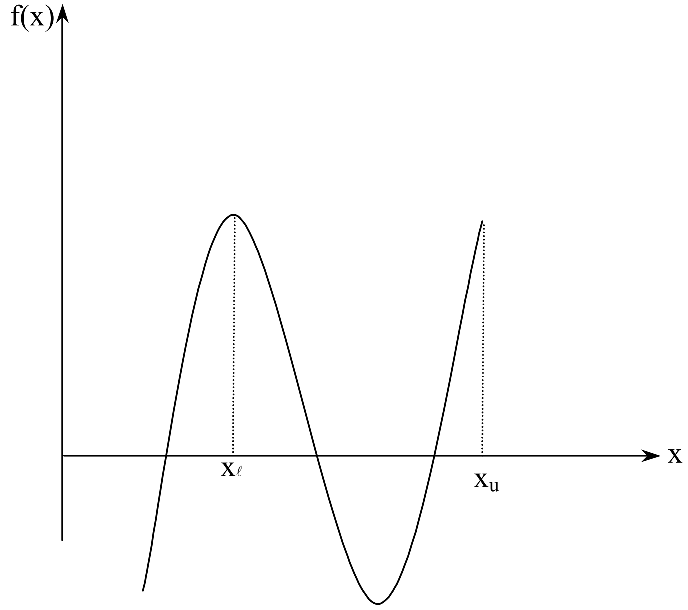
Basis of Bisection Method

Theorem: An equation f(x)=0, where f(x) is a real continuous function, has at least one root between x_ℓ and x_u if $f(x_l)\, f(x_u) < 0$.



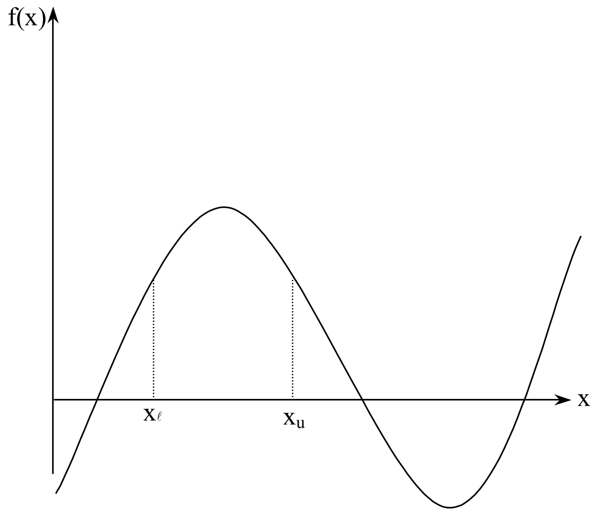
Theorem

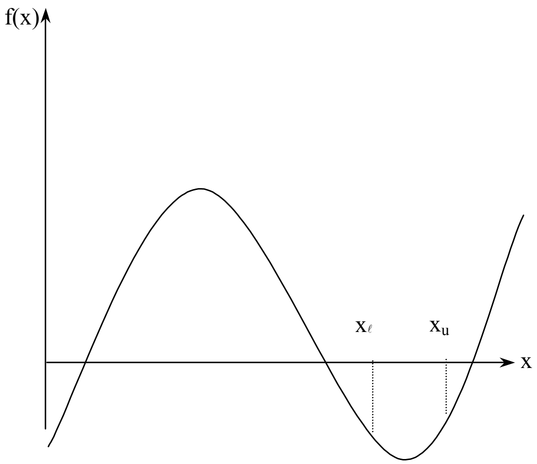
If function f(x) in f(x)=0 does not change sign between two points, roots may still exist between the two points.



Theorem

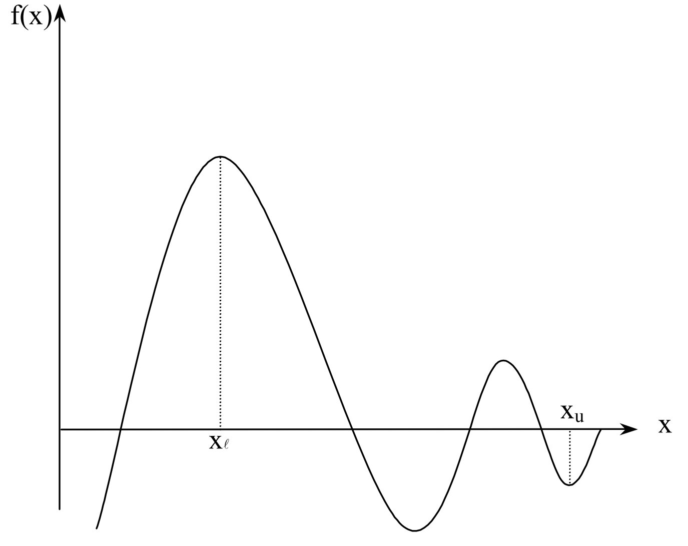
If the function f(x) in f(x)=0 does not change sign between two points, there may not be any roots between the two points.





Theorem

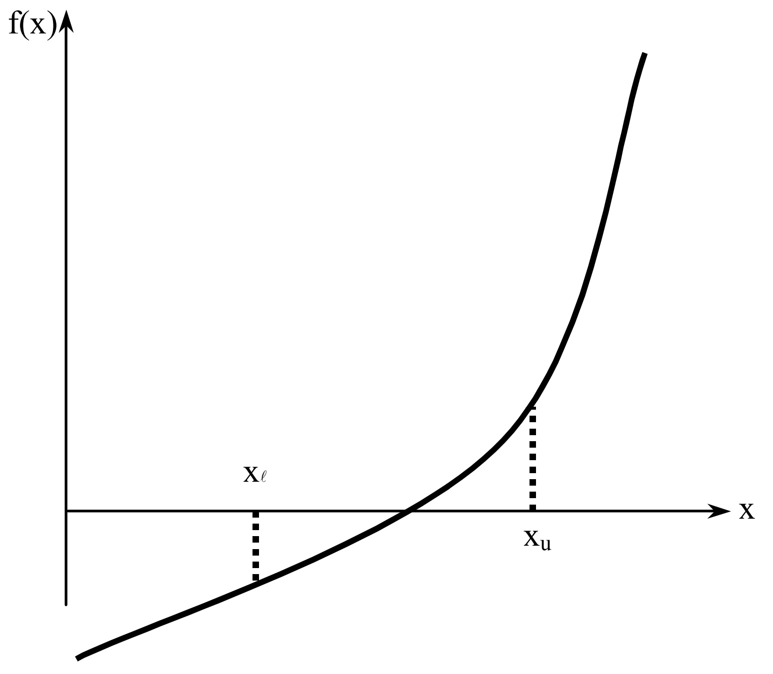
If the function f(x) in f(x)=0 changes sign between two points, more than one root may exist between the two points.



Algorithm for Bisection Method

Step 1

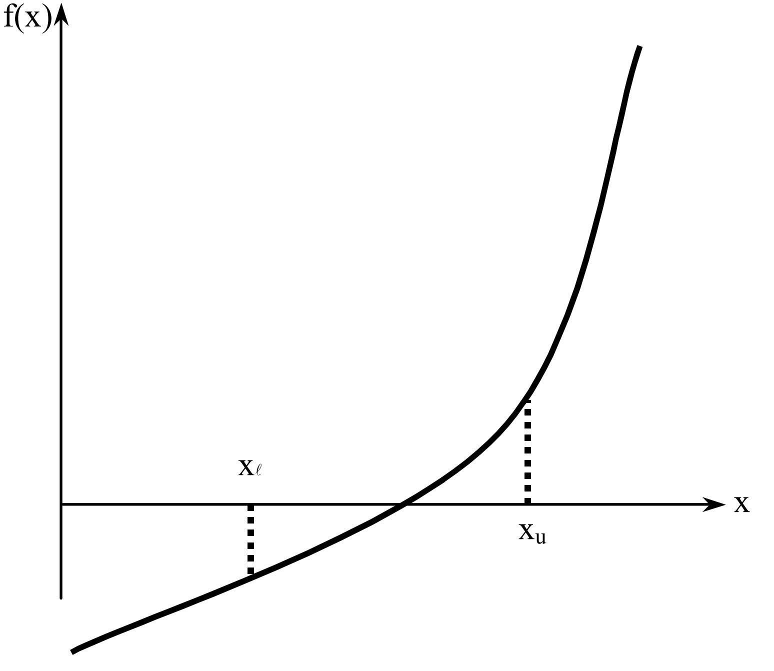
- Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)\, f(x_u) < 0$, or in other words, f(x) changes sign between x_ℓ and x_u.



Step 2

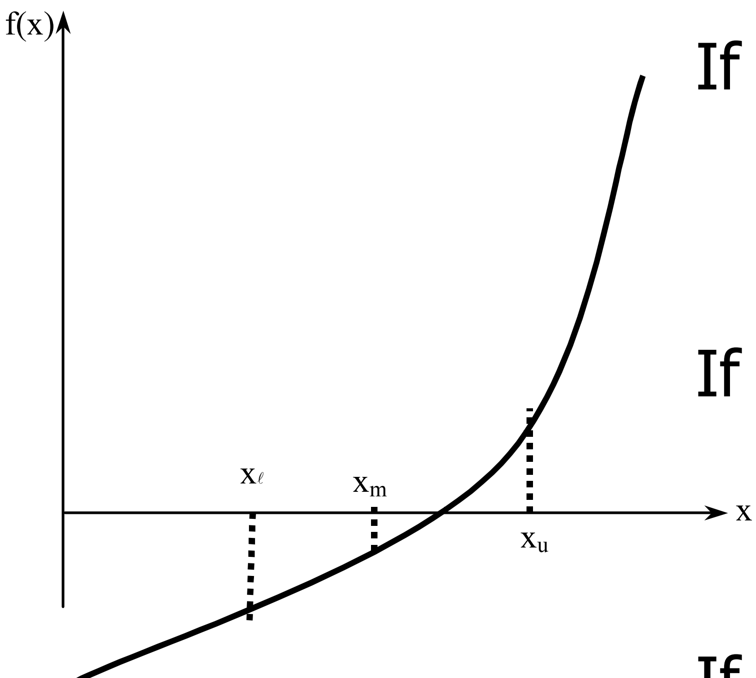
Estimate the root, x_m of the equation f (x) = 0 as the mid-point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$



Step 3

Now check the following



If $f(x_\ell)\, f(x_m) < 0$, then the root lies between x_ℓ and x_m; then $x_\ell = x_\ell$; $x_u = x_m$.

If $f(x_\ell)\, f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\ell = x_m$; $x_u = x_u$.

If $f(x_\ell)\, f(x_m) = 0$; then the root is x_m. Stop the algorithm if this is true.

Step 4

New estimate

$$x_m = \frac{x_\ell + x_u}{2}$$

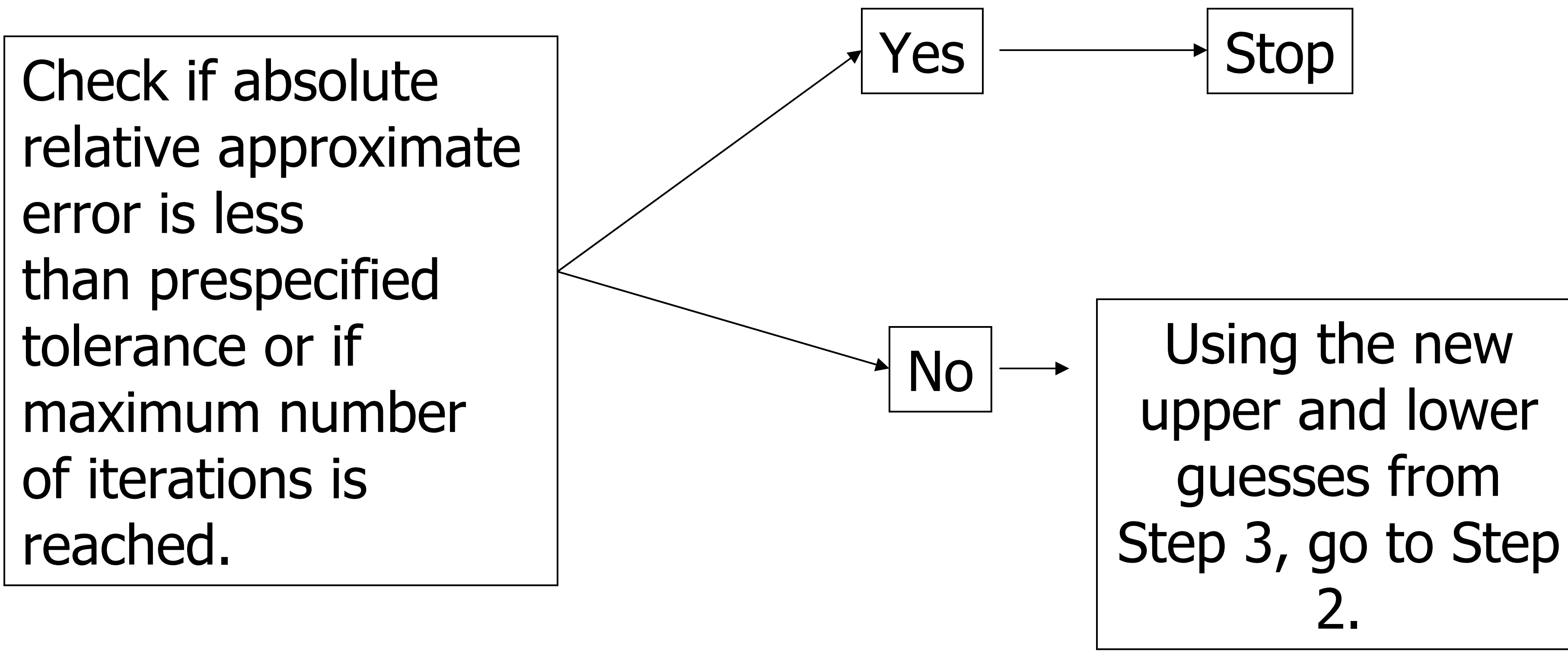
Absolute Relative Approximate Error

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

$$x_m^{old} = \text{previous estimate of root}$$

$$x_m^{new} = \text{current estimate of root}$$

Step 5


Check if absolute relative approximate error is less than prespecified tolerance or if maximum number of iterations is reached.
Yes
Stop
No
Using the new upper and lower guesses from Step 3, go to Step 2.

Example

- To find the inverse of a number 'a', one can use the equation

$$f(x) = ax - 1 = 0$$

where x is the inverse of 'a'.

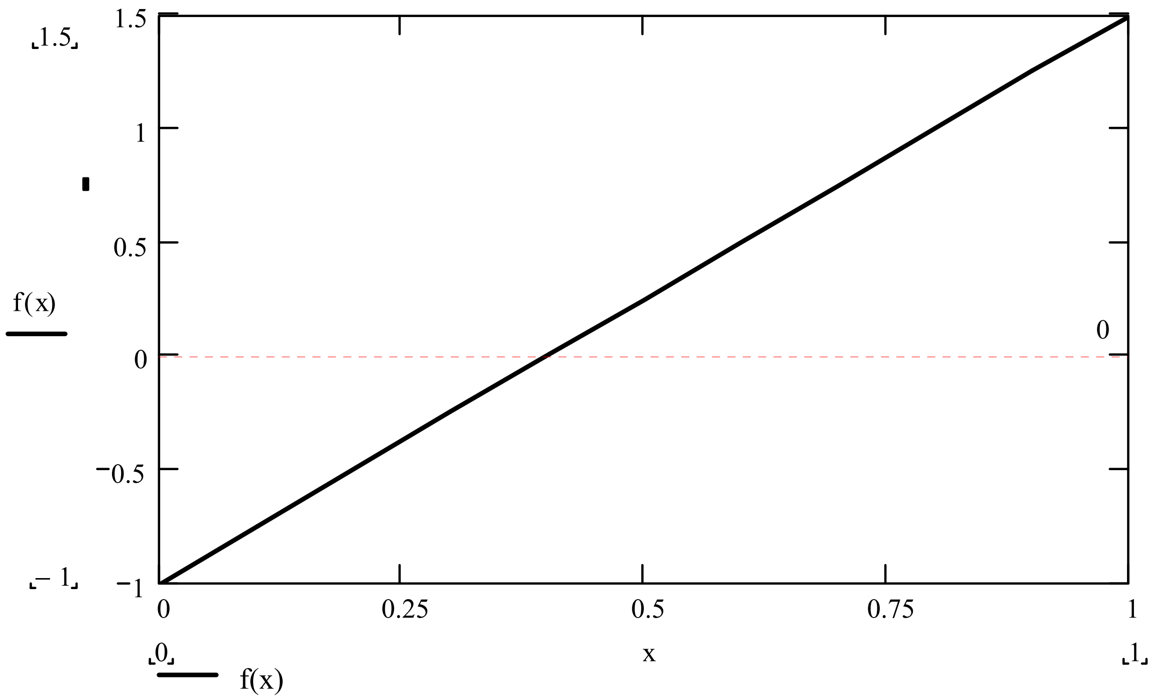
Solution

Use the bisection method of finding roots of equations to

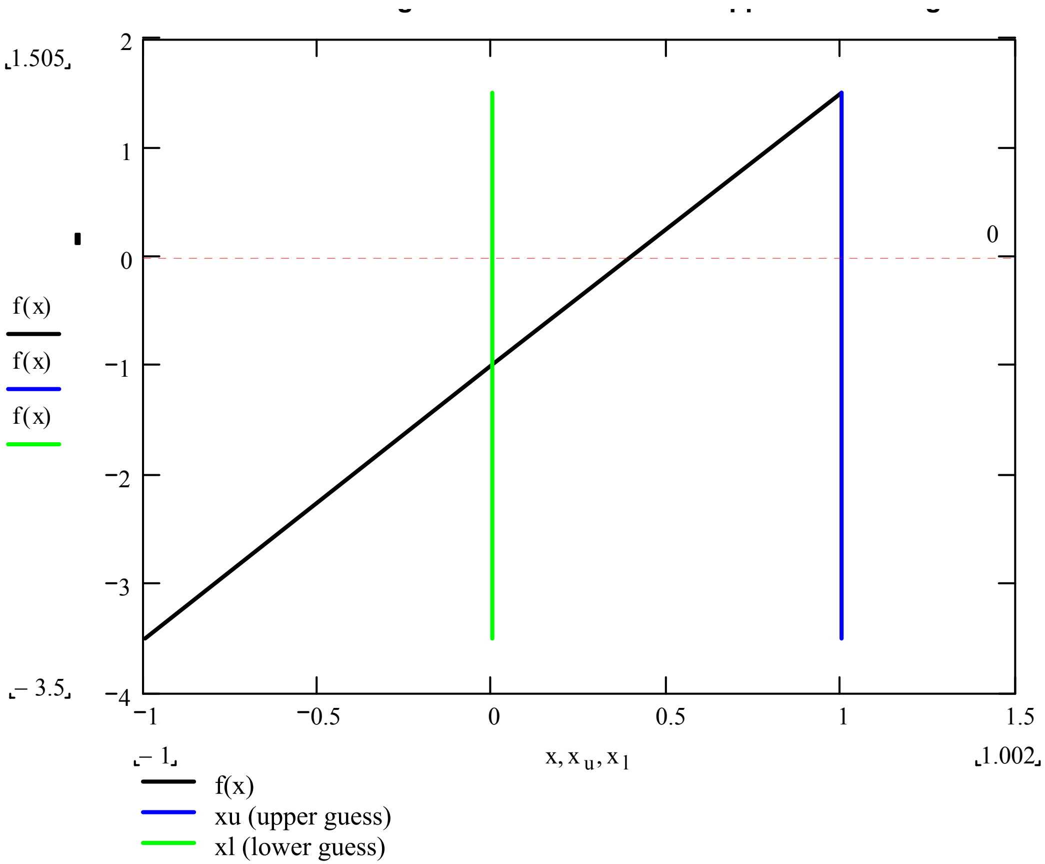
- Find the inverse of a = 2.5. Conduct three iterations to estimate the root of the above equation.
- Find the absolute relative approximate error at the end of each iteration, and
- The number of significant digits at least correct at the end of each iteration.

Graph of function f(x)

$$f(x) = ax - 1 = 0$$



Checking if the bracket is valid



Choose the bracket

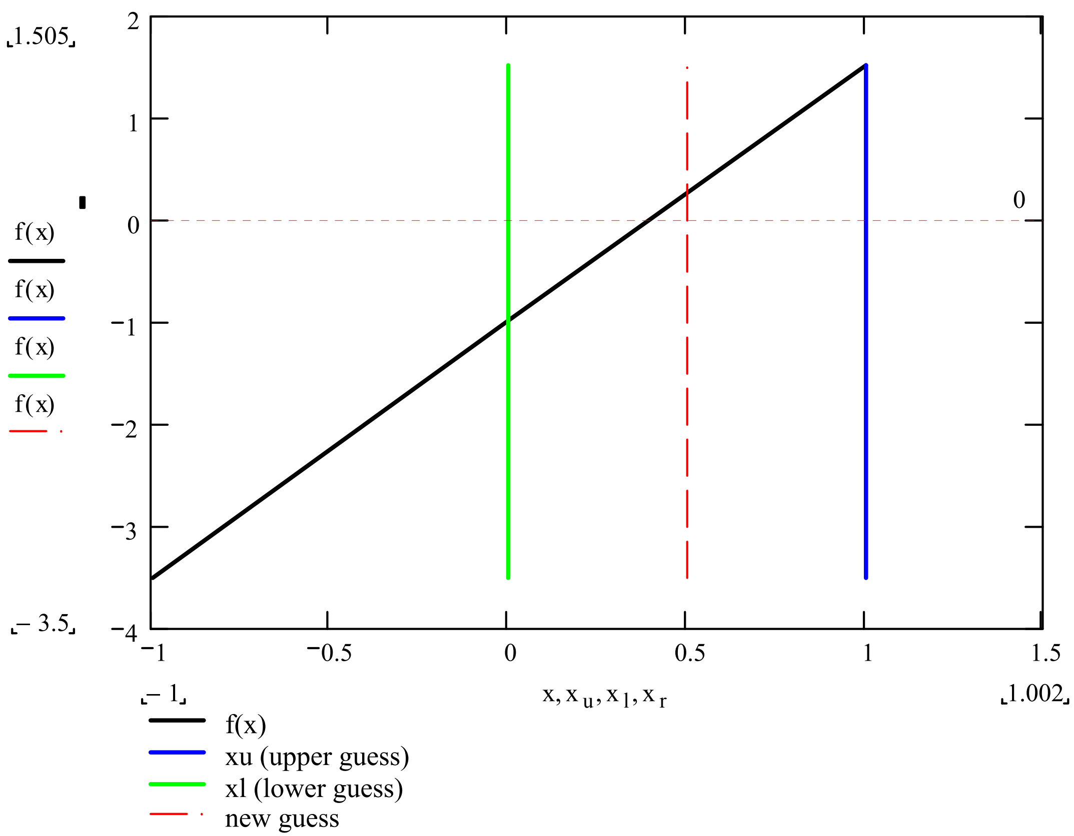
$$x_\ell = 0$$

$$x_u = 1$$

$$f(0) = -1$$

$$f(1) = 1.5$$

Iteration #1



$$x_\ell = 0, x_u = 1$$

$$x_m = \frac{0+1}{2} = 0.5$$

$$f(0) = -1$$

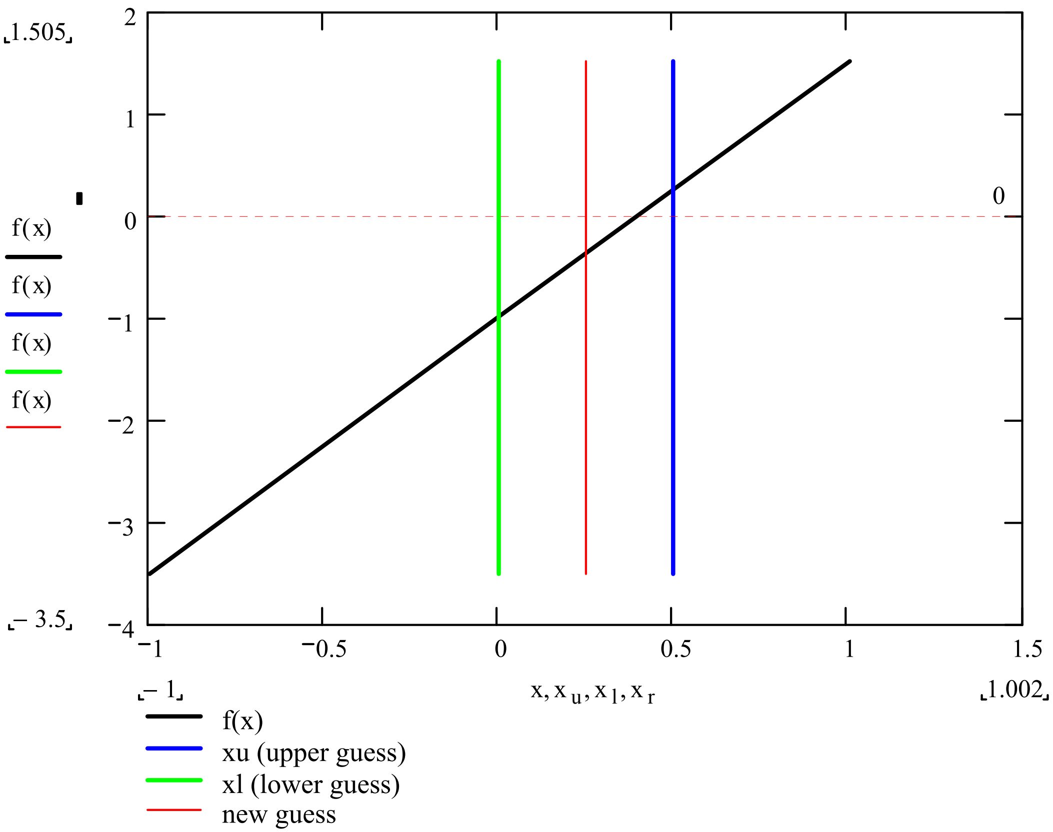
$$f(1) = 1.5$$

$$f(0.5) = 0.25$$

$$x_\ell = 0$$

$$x_u = 0.5$$

Iteration #2



$$x_\ell = 0, x_u = 0.5$$

$$x_m = \frac{0+0.5}{2} = 0.25$$

$$|\in_a| = 100\%$$

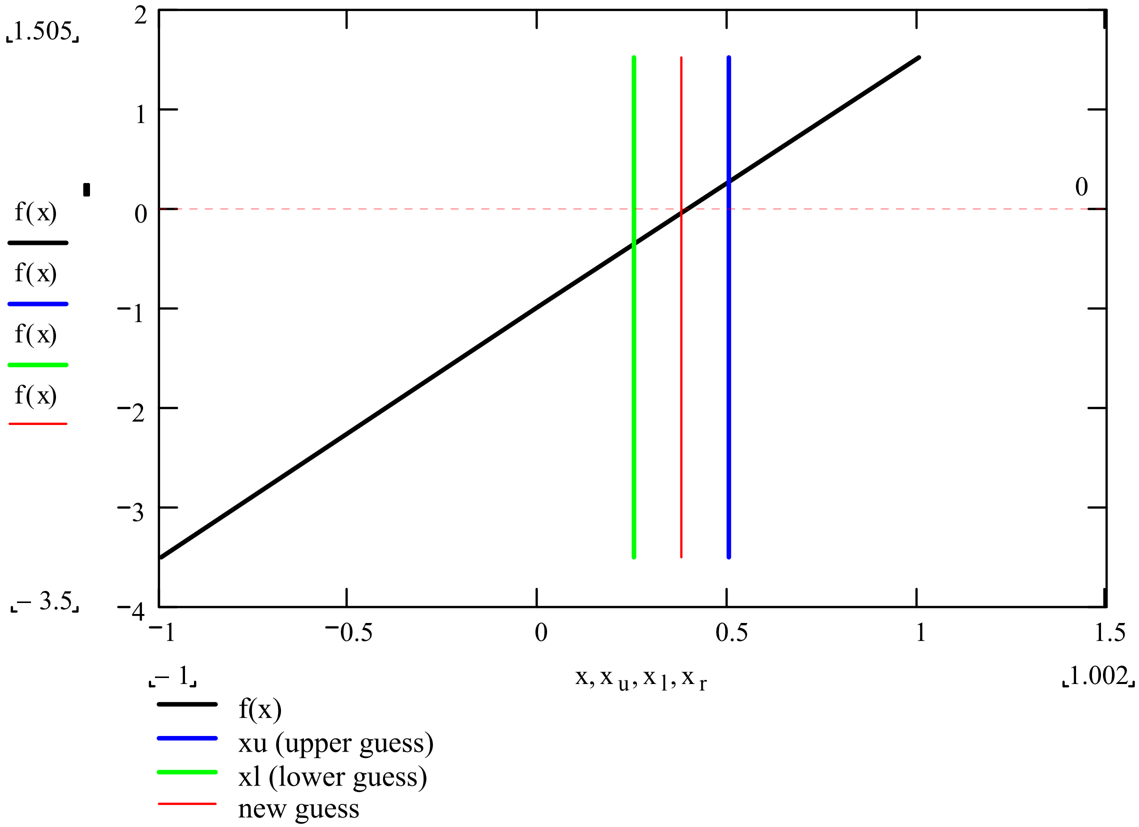
$$f(0) = -1$$

$$f(0.5) = 0.25$$

$$f(0.25) = -0.375$$

$$x_\ell = 0.25, x_u = 0.5$$

Iteration #3



$$x_\ell = 0.25, x_u = 0.5$$

$$x_m = \frac{0.25 + 0.5}{2} = 0.375$$

$$|\in_a| = 33.33\%$$

$$f(0.25) = -0.375$$

$$f(0.5) = 0.25$$

$$f(0.375) = -0.0625$$

$$x_\ell = 0.375, x_u = 0.500$$

Convergence

Table 1: Root of f(x)=0 as function of number of iterations for bisection method.

Iteration	x_l	x_u	x_m	$\lvert\epsilon_a\rvert$ %	$f(x_m)$
1	0	1	0.5	----------	0.25
2	0	0.5	0.25	100	-0.375
3	0.25	0.5	0.375	33.33	-0.0625
4	0.375	0.5	0.4375	14.2857	0.09375
5	0.375	0.4375	0.40625	7.6923	0.01563
6	0.375	0.40625	0.39063	4.00	-0.02344
7	0.39063	0.40625	0.39844	1.96078	$-3.90625\text{x}10^{-3}$
8	0.39844	0.40625	0.40234	0.97087	$5.8594\text{x}10^{-3}$
9	0.39844	0.40234	0.40039	0.4878	$9.7656\text{x}10^{-4}$
10	0.39844	0.40039	0.39941	0.2445	$-1.4648\text{x}10^{-3}$

Advantages

- Always convergent
- The root bracket gets halved with each iteration - guaranteed.

Drawbacks

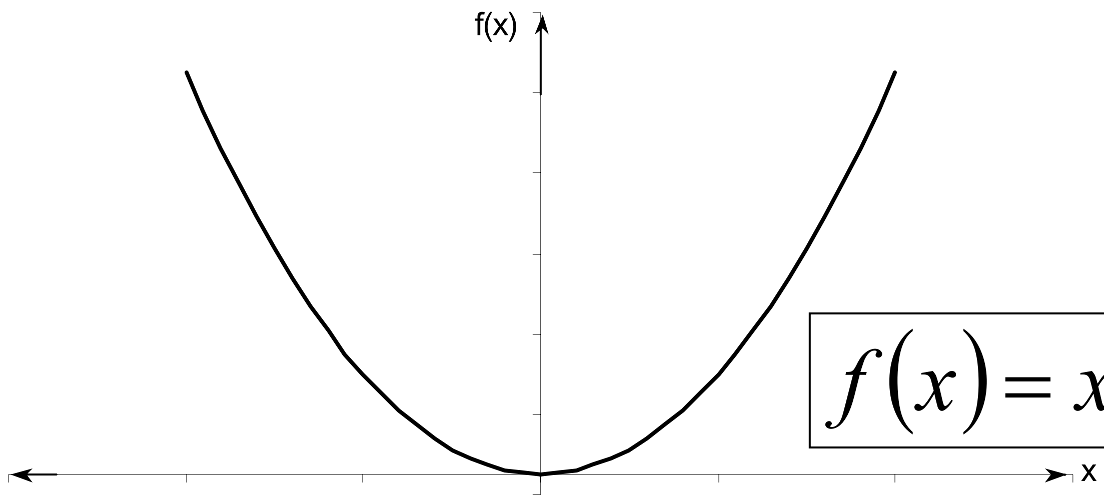
- Slow convergence

Drawbacks (continued)

- If one of the initial guesses is close to the root, the convergence is slower

Drawbacks (continued)

- If a function f(x) is such that it just touches the x-axis it will be unable to find the lower and upper guesses.



$$f(x) = x^2$$

Drawbacks (continued)

- Function changes sign but root does not exist

$$f(x) = \frac{1}{x}$$