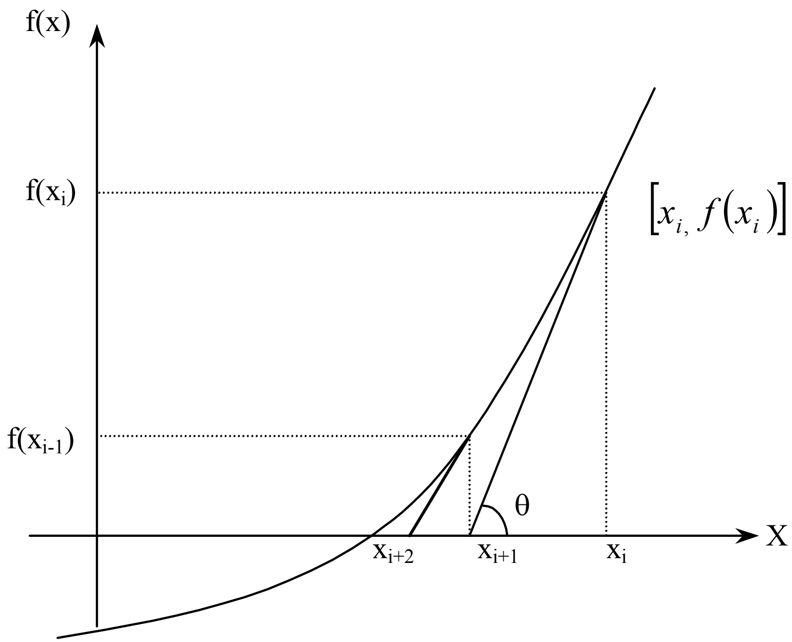
Roots of a Nonlinear Equation

Topic: Newton-Raphson Method

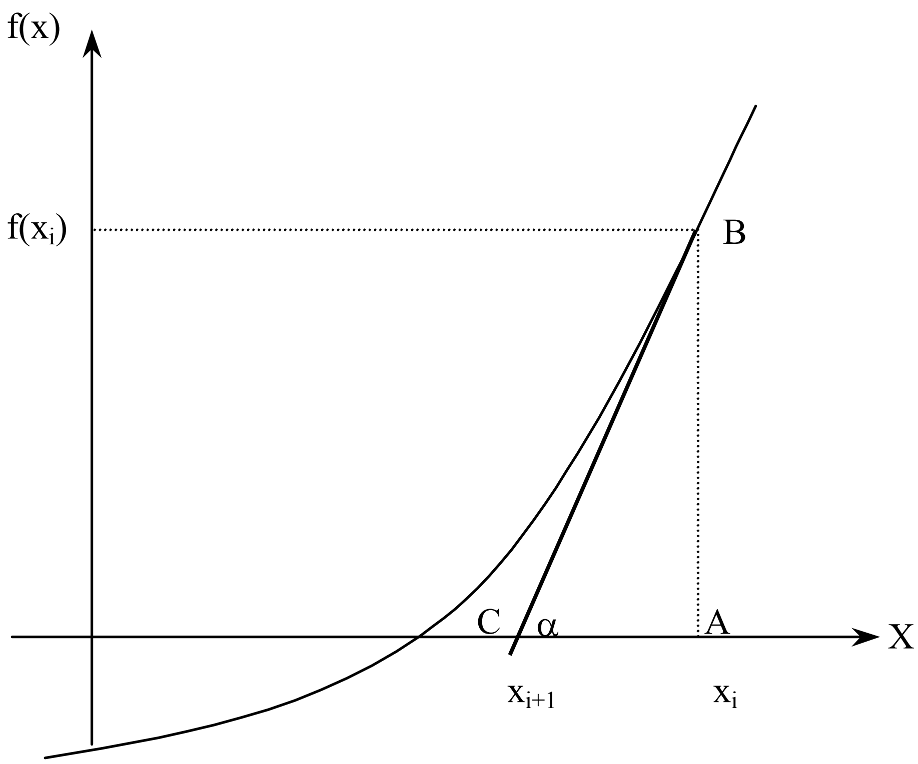
Major: Computer Engineering

Newton-Raphson Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



Derivation



$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Algorithm for Newton-Raphson Method

Step 1

Evaluate $f'(x)$ symbolically

Step 2

Calculate the next estimate of the root

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Find the absolute relative approximate error

$$\left|\in_a\right| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \text{x } 100$$

Step 3

- Find if the absolute relative approximate error is greater than the pre-specified relative error tolerance.
- If so, go back to step 2, else stop the algorithm.
- Also check if the number of iterations has exceeded the maximum number of iterations.

Example

- To find the inverse of a number 'a', one can use the equation

$$f(x) = a - \frac{1}{x} = 0$$

where x is the inverse of 'a'.

Solution

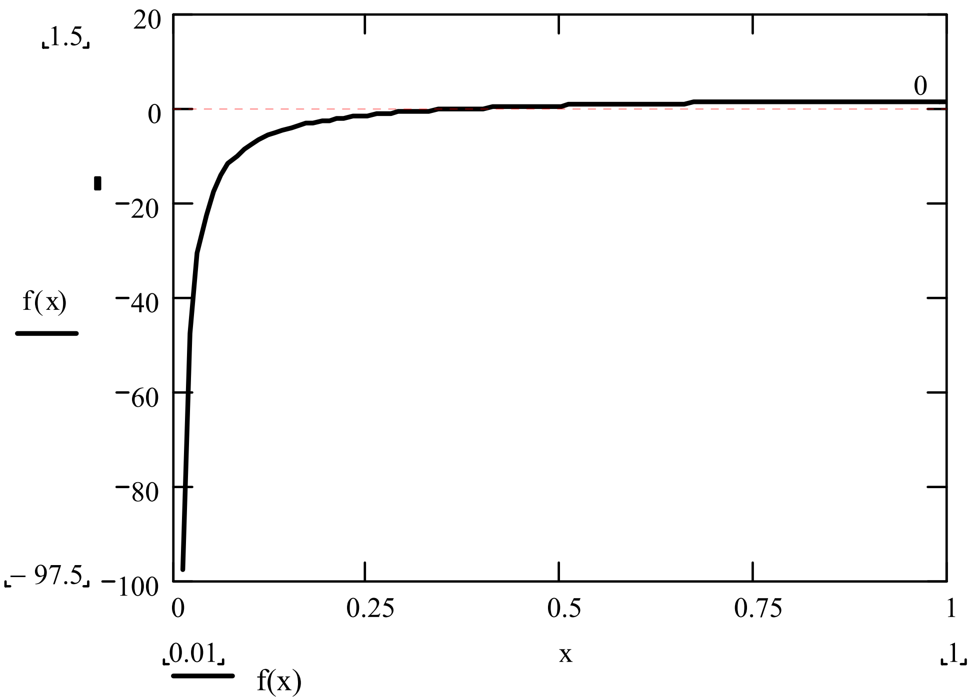
Use the Newton method of finding roots of equations to

- Find the inverse of a = 2.5. Conduct three iterations to estimate the root of the above equation.
- Find the absolute relative approximate error at the end of each iteration, and
- The number of significant digits at least correct at the end of each iteration.

Graph of function f(x)

$$f(x) = a - \frac{1}{x} = 0$$

$$f'(x) = \frac{1}{x^2}$$



$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

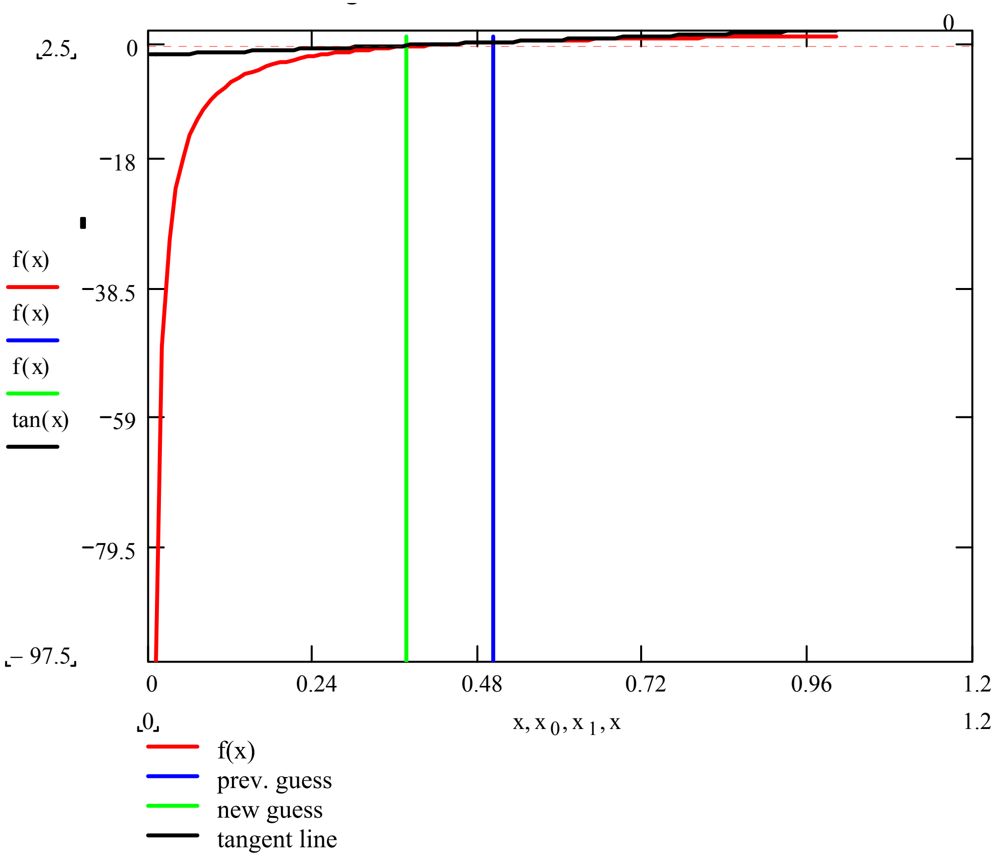
$$= x_i - \frac{a - \frac{1}{x_i}}{\frac{1}{x_i^2}}$$

$$= x_i - x_i^2\left(a - \frac{1}{x_i}\right)$$

$$= x_i - x_i^2 a + x_i$$

$$\therefore x_{i+1} = 2x_i - x_i^2 a$$

Iteration #1



$$x_0 = 0.5$$

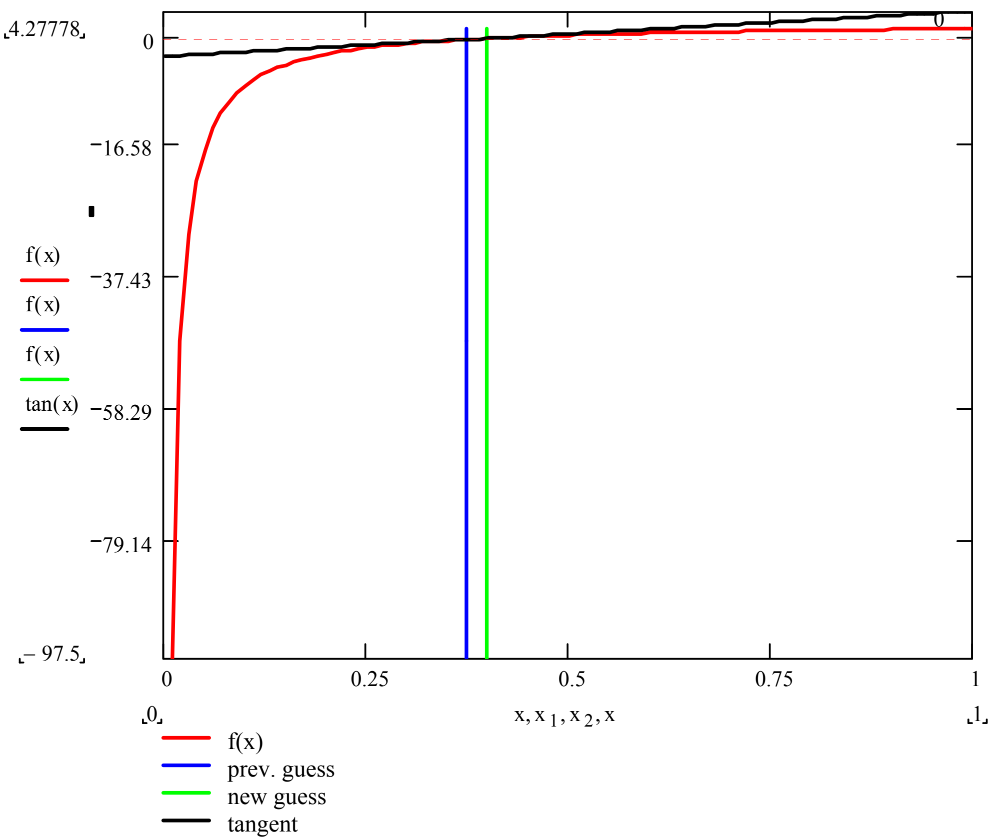
$$x_1 = 2x_0 - x_0^2 a$$

$$x_1 = 2(0.5) - (0.5)^2 2.5$$

$$= 0.375$$

$$|\in_a| = 33.33\%$$

Iteration #2



$$x_1 = 0.375$$

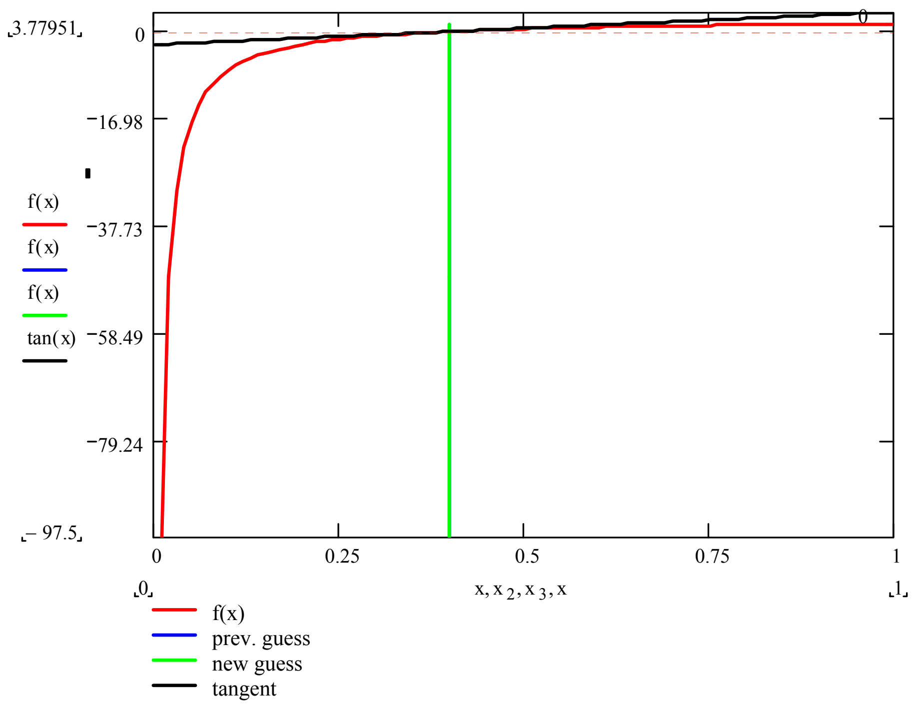
$$x_2 = 2x_1 - x_1^2 a$$

$$x_2 = 2(0.375) - (0.345)^2 2.5$$

$$= 0.3984$$

$$|\in_a| = 5.8224\ \%$$

Iteration #3



$$x_2 = 0.3984$$

$$x_3 = 2x_2 - x_2^2 a$$

$$x_3 = 2(0.3984) - (0.3984)^2 2.5$$

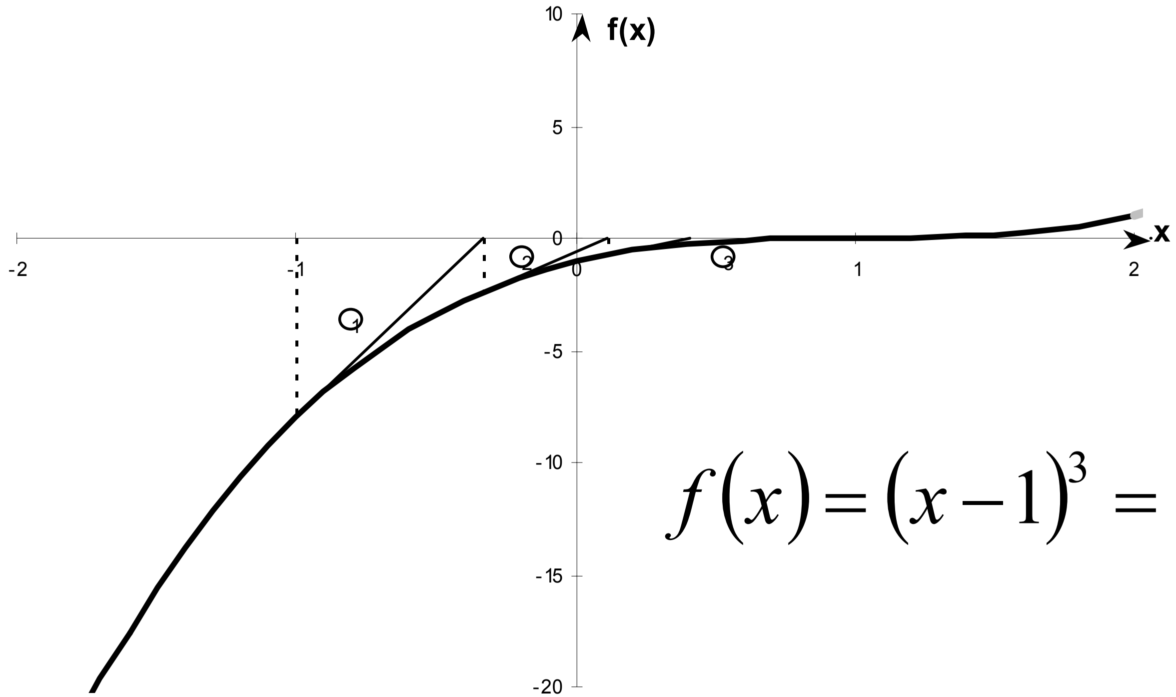
$$= 0.399$$

$$\left|\in_a\right| = 0.3891\%$$

Advantages

- Converges fast, if it converges
- Requires only one guess

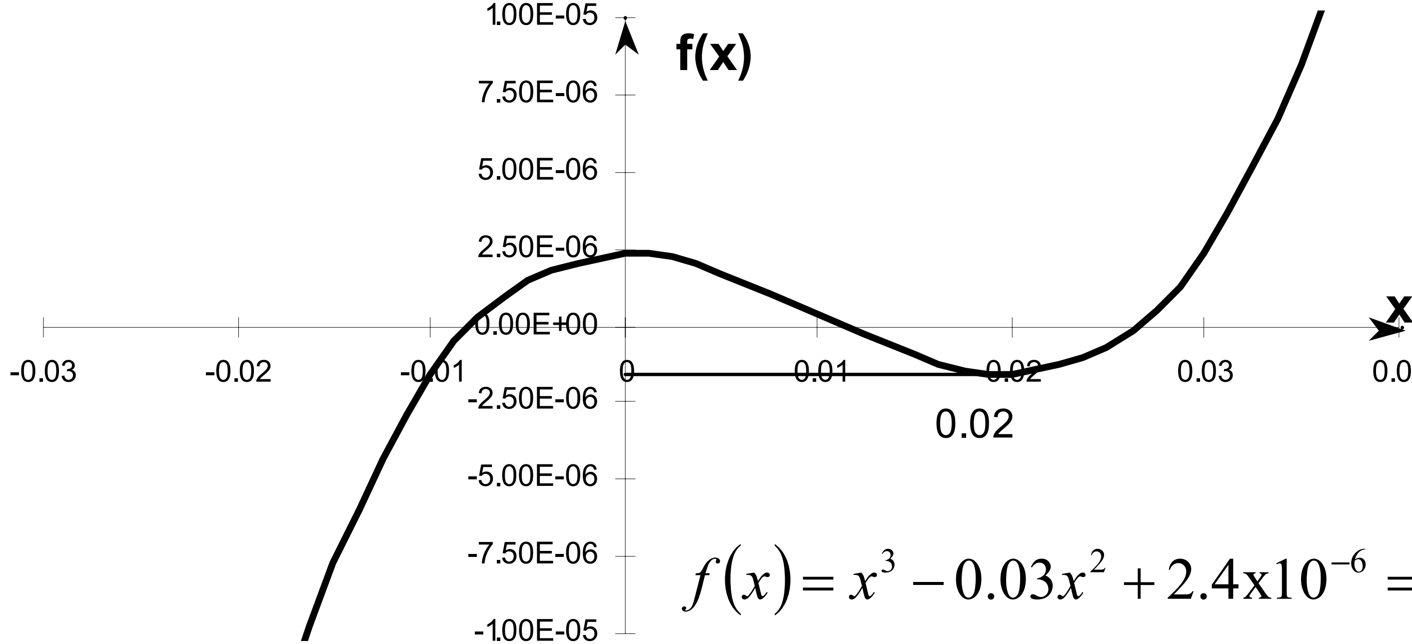
Drawbacks



$$f(x)=(x-1)^3=0$$

Inflection Point

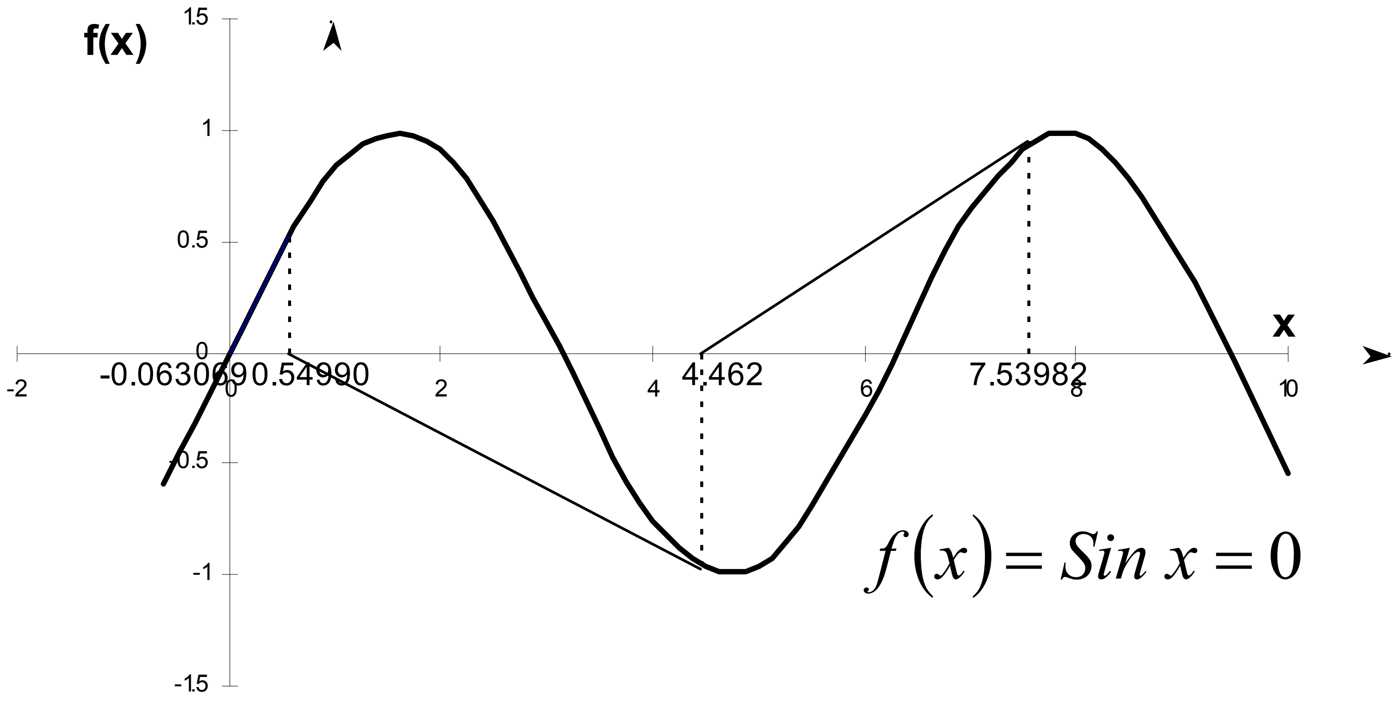
Drawbacks (continued)



$$f(x) = x^3 - 0.03x^2 + 2.4\text{x}10^{-6} = 0$$

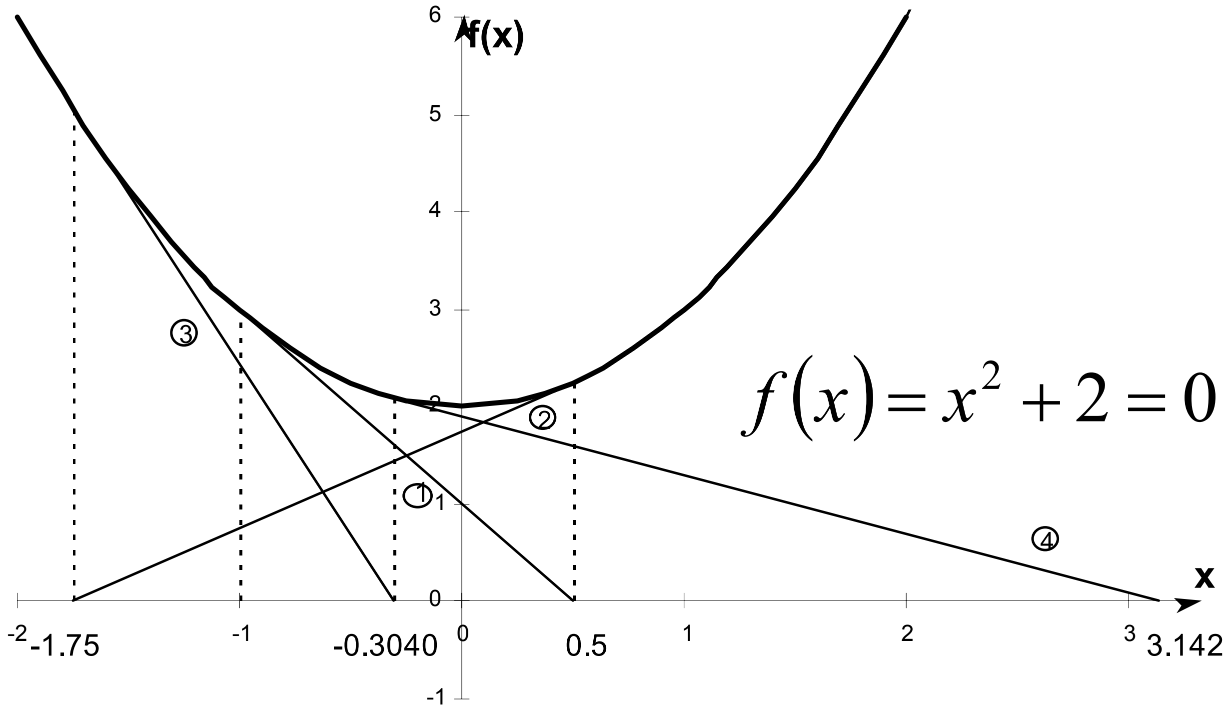
Division by zero

Drawbacks (continued)



Root Jumping

Drawbacks (continued)



Oscillations near Local Maxima or Minima