



Simultaneous Linear Equations



Topic: LU Decomposition
Major: Computer Engineering



LU Decomposition

LU Decomposition is another method to solve a set of simultaneous linear equations

Which is better, Gauss Elimination or LU Decomposition?

To answer this, a closer look at LU decomposition is needed.



LU Decomposition

Method

For most non-singular matrix $[A]$ that one could conduct Naïve Gauss Elimination forward elimination steps, one can always write it as

$$[A] = [L][U]$$

Where

$[L]$ = lower triangular matrix

$[U]$ = upper triangular matrix



LU Decomposition

Proof

If solving a set of linear equations $[A][X] = [C]$

If $[A] = [L][U]$ Then $[L][U][X] = [C]$

Multiply by $[L]^{-1}$

Which gives $[L]^{-1}[L][U][X] = [L]^{-1}[C]$

Remember $[L]^{-1}[L] = [I]$ which leads to $[I][U][X] = [L]^{-1}[C]$

Now, if $[I][U] = [U]$ then $[U][X] = [L]^{-1}[C]$

Now, let $[L]^{-1}[C] = [Z]$

Which ends with $[L][Z] = [C]$ (1)

and $[U][X] = [Z]$ (2)



LU Decomposition

How can this be used?

Given $[A][X]=[C]$

Decompose $[A]$ into $[L]$ and $[U]$

Then solve $[L][Z]=[C]$ for $[Z]$

And then solve $[U][X]=[Z]$ for $[X]$



LU Decomposition

How is this better or faster than Gauss Elimination?

Let's look at computational time.

n = number of equations

To decompose $[A]$, time is proportional to $\frac{n^3}{3}$

To solve $[U][X] = [C]$ and $[L][Z] = [C]$

time proportional to $\frac{n^2}{2}$



LU Decomposition

Therefore, total computational time for LU Decomposition is proportional to

$$\frac{n^3}{3} + 2\left(\frac{n^2}{2}\right) \quad \text{or} \quad \frac{n^3}{3} + n^2$$

Gauss Elimination computation time is proportional to

$$\frac{n^3}{3} + \frac{n^2}{2}$$

How is this better?



LU Decomposition

What about a situation where the [C] vector changes?

In LU Decomposition, LU decomposition of [A] is independent of the [C] vector, therefore it only needs to be done once.

Let m = the number of times the [C] vector changes

The computational times are proportional to

$$\text{LU decomposition} = m\left(\frac{n^3}{3} + \frac{n^2}{2}\right) \quad \text{Gauss Elimination} = \frac{n^3}{3} + m(n^2)$$

Consider a 100 equation set with 50 right hand side vectors

$$\text{LU Decomposition} = 8.33 \times 10^5 \quad \text{Gauss Elimination} = 1.69 \times 10^7$$



LU Decomposition

Another Advantage

Finding the Inverse of a Matrix

LU Decomposition

$$\frac{n^3}{3} + n(n^2) = \frac{4n^3}{3}$$

Gauss Elimination

$$n \left(\frac{n^3}{3} + \frac{n^2}{2} \right) = \frac{n^4}{3} + \frac{n^3}{2}$$

For large values of n

$$\frac{n^4}{3} + \frac{n^3}{2} \gg \frac{4n^3}{3}$$



LU Decomposition

Method: [A] Decompose to [L] and [U]

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

[U] is the same as the coefficient matrix at the end of the forward elimination step.

[L] is obtained using the *multipliers* that were used in the forward elimination process

Example: Surface Shape Detection

To infer the surface shape of an object from images taken of a surface from three different directions, one needs to solve the following set of equations

$$\begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ -0.2357 & -0.2357 & -0.9428 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 247 \\ 248 \\ 239 \end{bmatrix}$$

The right hand side are the light intensities from the middle of the images, while the coefficient matrix is dependent on the light source directions with respect to the camera. The unknowns are the incident intensities that will determine the shape of the object.

Example: Surface Shape Detection

The solution for the unknowns x_1 , x_2 , and x_3 is given by

$$\begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ -0.2357 & -0.2357 & -0.9428 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 247 \\ 248 \\ 239 \end{bmatrix}$$

Find the values of x_1 , x_2 , and x_3 using LU Decomposition.

Example: Surface Shape Detection

Finding the $[U]$ matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ -0.2357 & -0.2357 & -0.9428 \end{bmatrix}$$

$$\text{Row2} - \left[\frac{\text{Row1}}{0.2425} \right] \times (0) = \begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ -0.2357 & -0.2357 & -0.9428 \end{bmatrix}$$

$$\text{Row3} - \left[\frac{\text{Row1}}{0.2425} \right] \times (-0.2357) = \begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ 0 & -0.2357 & -1.8860 \end{bmatrix}$$

Example: Surface Shape Detection

Finding the $[U]$ matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ 0 & -0.2357 & -1.8860 \end{bmatrix}$$

$$\text{Row2} - \left[\frac{\text{Row2}}{0.2425} \right] \times (-0.2357) = \begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ 0 & 0 & -2.8286 \end{bmatrix}$$

$$[U] = \begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ 0 & 0 & -2.8286 \end{bmatrix}$$

Example: Surface Shape Detection

Finding the $[L]$ matrix

Using the multipliers used during the Forward Elimination Procedure

From the first step
of forward
elimination

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$l_{21} = \frac{a_{21}}{a_{11}} = \frac{0}{0.2425} = 0$$

$$l_{31} = \frac{a_{31}}{a_{11}} = \frac{-0.2357}{0.2425} = -0.9720$$

From the second
step of forward
elimination

$$\begin{bmatrix} 4 & 2 & 1 \\ 0 & -0.625 & -0.5625 \\ 0 & -1.5 & -1.25 \end{bmatrix}$$

$$l_{32} = \frac{a_{32}}{a_{22}} = \frac{-0.2357}{0.2425} = -0.9720$$

Example: Surface Shape Detection

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.9720 & -0.9720 & 1 \end{bmatrix}$$

Does $[L][U] = [A]$?

$$[L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.9720 & -0.9720 & 1 \end{bmatrix} \begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ 0 & 0 & -2.8286 \end{bmatrix}$$

Example: Surface Shape Detection

Example: Solving simultaneous linear equations using LU Decomposition

$$\text{Set } [L][Z] = [C] \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.9720 & -0.9720 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 247 \\ 248 \\ 239 \end{bmatrix}$$

$$z_1 = 247$$

Solve for $[Z]$

$$0z_1 + z_2 = 248$$

$$(-0.9720)z_1 + (-0.9720)z_2 + z_3 = 239$$

Example: Surface Shape Detection

Example: Solving simultaneous linear equations using LU Decomposition

Complete the forward substitution to solve for $[Z]$

$$z_1 = 247$$

$$\begin{aligned} z_2 &= 248 - 0z_1 \\ &= 248 \end{aligned}$$

$$\begin{aligned} z_3 &= 239 - (-0.9720)z_1 - (-0.9720)z_2 \\ &= 239 - (-0.9720) \times 247 - (-0.9720) \times 248 \\ &= 720.1196 \end{aligned}$$

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 247 \\ 248 \\ 720.1196 \end{bmatrix}$$

Example: Surface Shape Detection

Example: Solving simultaneous linear equations using LU Decomposition

$$\text{Set } [U][X] = [Z] \quad \begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ 0 & 0 & -2.8286 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 247 \\ 248 \\ 720.1196 \end{bmatrix}$$

The 3 equations become

$$\begin{aligned} \text{Solve for } [X] \quad & 0.2425x_1 + 0x_2 + (-0.9701)x_3 = 247 \\ & 0.2425x_2 + (-0.9701)x_3 = 248 \\ & -2.8286x_3 = 720.1196 \end{aligned}$$

Example: Surface Shape Detection

Example: Solving simultaneous linear equations using LU Decomposition

From the 3rd equation

$$-2.8286x_3 = 720.1196$$

$$= \frac{720.1196}{-2.8286}$$

$$= -254.5857$$

Substituting in x_3 and using the second equation

$$0.2425x_2 + (-0.9701)x_3 = 248$$

$$x_2 = \frac{248 - (-0.9701)x_3}{0.2425}$$

$$= \frac{248 - (-0.9701) \times (-254.5857)}{0.2425}$$

$$= 4.2328$$

Example: Surface Shape Detection

Example: Solving simultaneous linear equations using LU Decomposition

Substituting in x_3 and x_2
using the first equation

$$0.2425x_1 + 0x_2 + (-0.9701)x_3 = 247$$

$$\begin{aligned}x_1 &= \frac{247 - 0x_2 - (-0.9701)x_3}{0.2425} \\ &= \frac{247 - (-0.9701) \times (-254.5857)}{0.2425} \\ &= 0.1091\end{aligned}$$

Hence the Solution Vector is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.1091 \\ 4.2328 \\ -254.5857 \end{bmatrix}$$



LU Decomposition

Finding the inverse of a square matrix

Remember, the relative computational time comparison of LU decomposition and Gauss elimination is:

$$\frac{n^4}{3} + \frac{n^3}{2} \gg \frac{4n^3}{3}$$

Review: The inverse $[B]$ of a square matrix $[A]$ is defined as

$$[A][B] = [I] = [B][A]$$



LU Decomposition

Finding the inverse of a square matrix

How can LU Decomposition be used to find the inverse?

Assume the first column of $[B]$ to be $[b_{11} \ b_{12} \ \dots \ b_{n1}]^T$

Using this and the definition of matrix multiplication

First column of $[B]$

$$[A] \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Second column of $[B]$

$$[A] \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

The remaining columns in $[B]$ can be found in the same manner



LU Decomposition

Example: Finding the inverse of a square matrix

Find the inverse of $[A]$

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Using the Decomposition procedure, the $[L]$ and $[U]$ matrices are found to be

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$



LU Decomposition

Example: Finding the inverse of a square matrix

Solving for the each column of $[B]$ requires to steps

1) Solve $[L][Z] = [C]$ for $[Z]$ and 2) Solve $[U][X] = [Z]$ for $[X]$

$$\text{Step 1: } [L][Z] = [C] \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This generates the equations:

$$z_1 = 1$$

$$2.56z_1 + z_2 = 0$$

$$5.76z_1 + 3.5z_2 + z_3 = 0$$



LU Decomposition

Example: Finding the inverse of a square matrix

Solving for $[Z]$

$$z_1 = 1$$

$$\begin{aligned} z_2 &= 0 - 2.56z_1 \\ &= 0 - 2.56(1) \\ &= -2.56 \end{aligned}$$

$$\begin{aligned} z_3 &= 0 - 5.76z_1 - 3.5z_2 \\ &= 0 - 5.76(1) - 3.5(-2.56) \\ &= 3.2 \end{aligned}$$

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$



LU Decomposition

Example: Finding the inverse of a square matrix

Solving for $[U] [X] = [Z]$ for $[X]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$25b_{11} + 5b_{21} + b_{31} = 1$$

$$-4.8b_{21} - 1.56b_{31} = -2.56$$

$$0.7b_{31} = 3.2$$



LU Decomposition

Example: Finding the inverse of a square matrix

Using Backward Substitution

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$\begin{aligned} b_{21} &= \frac{-2.56 + 1.560b_{31}}{-4.8} \\ &= \frac{-2.56 + 1.560(4.571)}{-4.8} = -0.9524 \end{aligned}$$

$$\begin{aligned} b_{11} &= \frac{1 - 5b_{21} - b_{31}}{25} \\ &= \frac{1 - 5(-0.9524) - 4.571}{25} = 0.04762 \end{aligned}$$

So the first column of the inverse of $[A]$ is:

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

LU Decomposition

Example: Finding the inverse of a square matrix

Repeating for the second and third columns of the inverse

Second Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

Third Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$



LU Decomposition

Example: Finding the inverse of a square matrix

The inverse of $[A]$ is

$$[A]^{-1} = \begin{bmatrix} 0.4762 & 0.08333 & 0.0357 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.050 & 1.429 \end{bmatrix}$$

To check your work do the following operation

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$