

Interpolation



Topic: Direct Method

Major: Computer Science

What is Interpolation ?

Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, find the value of 'y' at a value of 'x' that is not given.





Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.



Direct Method

Given ' $n+1$ ' data points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, pass a polynomial of order ' n ' through the data as given below:

$$y = a_0 + a_1x + \dots + a_nx^n.$$

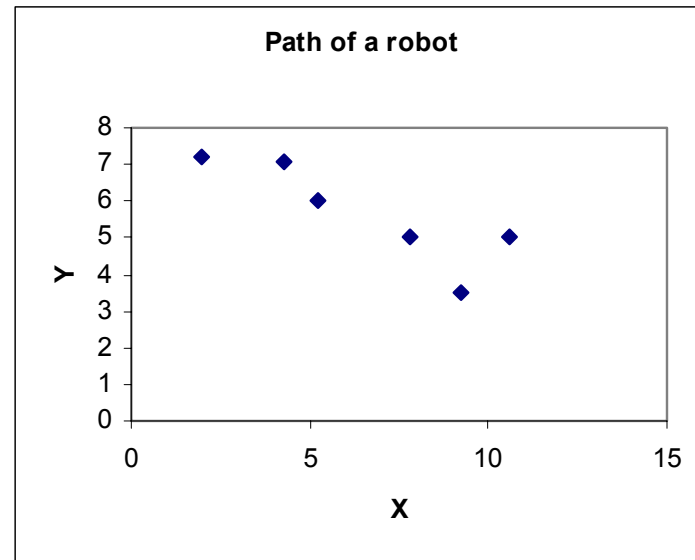
where a_0, a_1, \dots, a_n are real constants.

- Set up ' $n+1$ ' equations to find ' $n+1$ ' constants.
- To find the value ' y ' at a given value of ' x ', simply substitute the value of ' x ' in the above polynomial.

Example

The path of a rapid laser is given by these specifications. If the laser is traversing from $x = 2$ to $x = 4.25$ in a linear path, find the value of y at $x = 4$ using the direct method.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6
7.81	5
9.2	3.5
10.6	5



Linear Interpolation

$$y(x) = a_0 + a_1 x$$

$$y(2.00) = a_0 + a_1(2.00) = 7.2$$

$$y(4.25) = a_0 + a_1(4.25) = 7.1$$

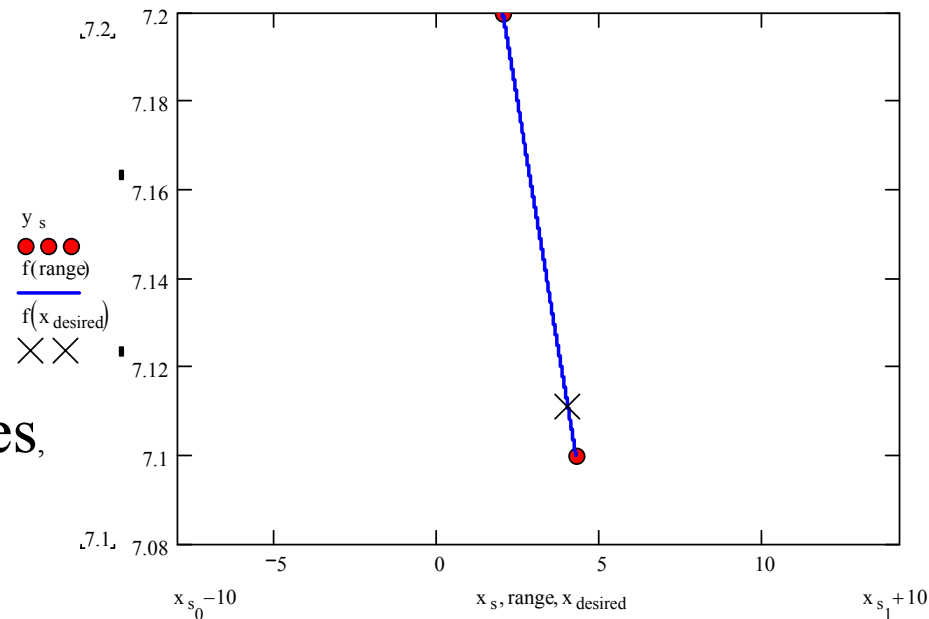
Solving the above two equations gives,

$$a_0 = 7.2889 \quad a_1 = -0.04444$$

Hence

$$y(x) = 7.2889 - 0.04444x, \quad 2.00 \leq x \leq 4.25.$$

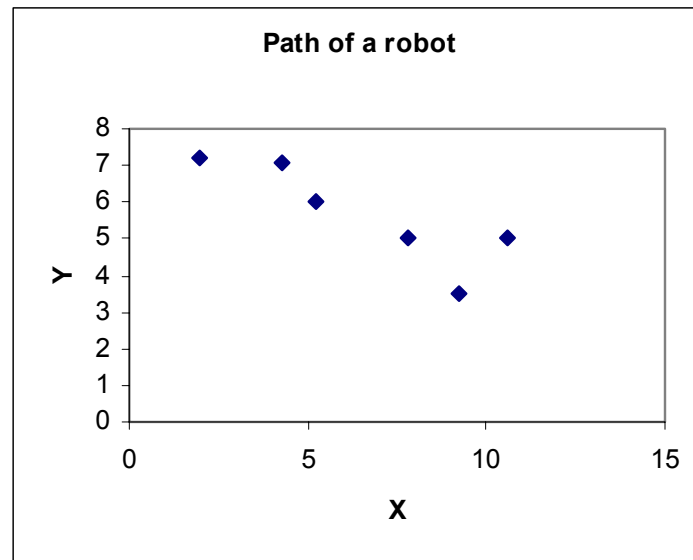
$$y(4.00) = 7.2889 - 0.04444(4.00) = 7.1111 \text{ in.}$$



Example

The path of a rapid laser is given by these specifications. If the laser is traversing from $x = 2$ to $x = 4.25$ to $x=5.25$ using a quadratic path, find the value of y at $x = 4$ using the direct method.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6
7.81	5
9.2	3.5
10.6	5





Quadratic Interpolation

$$y(x) = a_0 + a_1x + a_2x^2$$

$$y(2.00) = a_0 + a_1(2.00) + a_2(2.00)^2 = 7.2$$

$$y(4.25) = a_0 + a_1(4.25) + a_2(4.25)^2 = 7.1$$

$$y(5.25) = a_0 + a_1(5.25) + a_2(5.25)^2 = 6.0$$

Solving the above three equations gives

$$a_0 = 4.5282 \quad a_1 = 1.9855 \quad a_2 = -0.32479$$

Quadratic Interpolation (contd)

$$y(x) = 4.5282 + 1.9855x - 0.32479x^2, \quad 2.00 \leq x \leq 5.25$$

$$y(4.00) = 4.5282 + 1.9855(4.00) - 0.32479(4.00)^2$$

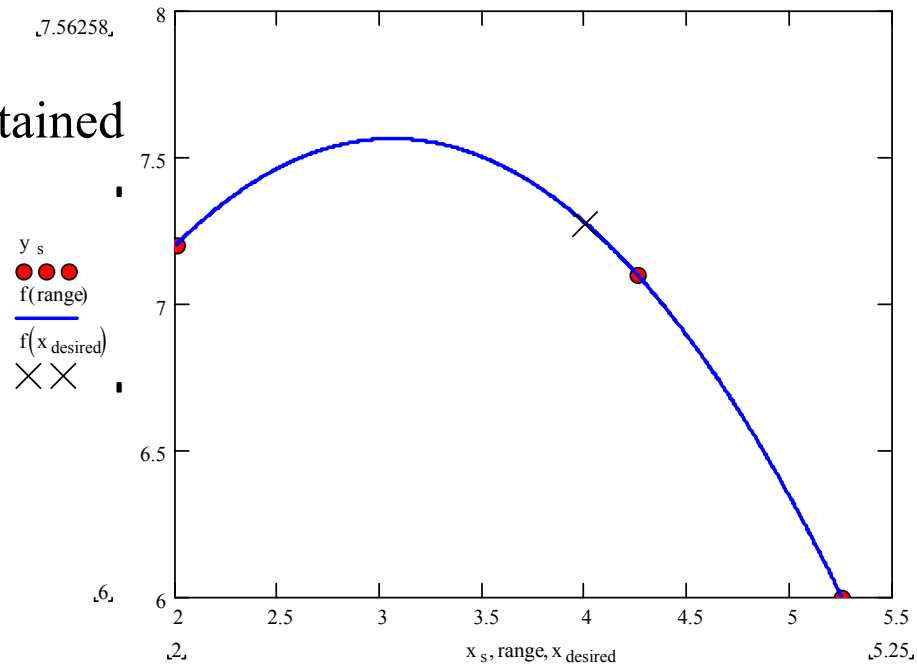
$$= 7.2735 \text{ in.}$$

The absolute relative approximate error $|\epsilon_a|$ obtained

between first and second order polynomial is

$$|\epsilon_a| = \left| \frac{7.2735 - 7.1111}{7.2735} \right| \times 100$$

$$= 2.2327\%$$





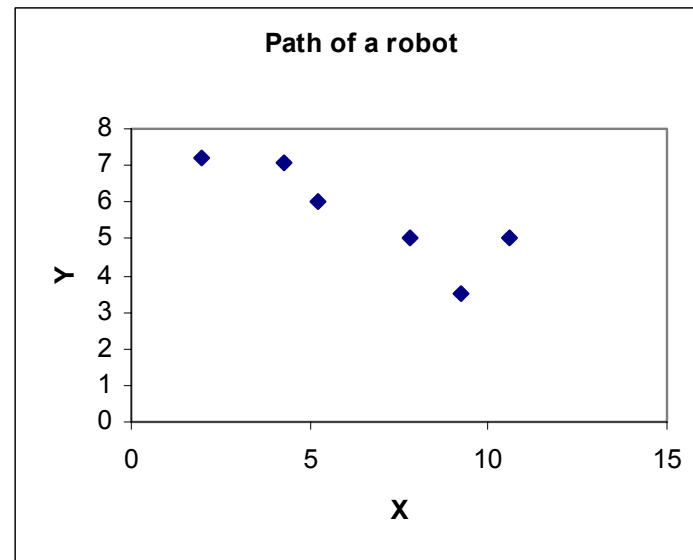
Comparison Table

Order of Polynomial	1	2
Temperature $^{\circ}C$	35.809	35.089
Absolute Relative Approximate Error	-----	2.0544%

Example

The path of a rapid laser is given by these specifications. Find the path traversed by the laser using the direct method. Use a fifth order polynomial.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6
7.81	5
9.2	3.5
10.6	5





Fifth Order Interpolation

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$$

$$y(2.00) = 7.2 = a_0 + a_1(2.00) + a_2(2.00)^2 + a_3(2.00)^3 + a_4(2.00)^4 + a_5(2.00)^5$$

$$y(4.25) = 7.1 = a_0 + a_1(4.25) + a_2(4.25)^2 + a_3(4.25)^3 + a_4(4.25)^4 + a_5(4.25)^5$$

$$y(5.25) = 6.0 = a_0 + a_1(5.25) + a_2(5.25)^2 + a_3(5.25)^3 + a_4(5.25)^4 + a_5(5.25)^5$$

$$y(7.81) = 5.0 = a_0 + a_1(7.81) + a_2(7.81)^2 + a_3(7.81)^3 + a_4(7.81)^4 + a_5(7.81)^5$$

$$y(9.20) = 3.5 = a_0 + a_1(9.20) + a_2(9.20)^2 + a_3(9.20)^3 + a_4(9.20)^4 + a_5(9.20)^5$$

$$y(10.60) = 5.0 = a_0 + a_1(10.60) + a_2(10.60)^2 + a_3(10.60)^3 + a_4(10.60)^4 + a_5(10.60)^5$$



Fifth Order Interpolation (contd)

Writing the six equations in matrix form, we have

$$\begin{bmatrix} 1 & 2.00 & 4.00 & 8.00 & 16.00 & 32 \\ 1 & 4.25 & 18.063 & 76.766 & 326.25 & 1386.6 \\ 1 & 5.25 & 27.563 & 144.70 & 759.69 & 3988.4 \\ 1 & 7.81 & 60.996 & 476.38 & 3720.5 & 29057 \\ 1 & 9.20 & 84.640 & 778.69 & 7163.9 & 65908 \\ 1 & 10.6 & 112.36 & 1191.0 & 12625 & 133820 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 7.2 \\ 7.1 \\ 6.0 \\ 5.0 \\ 3.5 \\ 5.0 \end{bmatrix}$$

$$a_0 = -30.898 \quad a_1 = 41.344 \quad a_2 = -15.855$$

$$a_3 = 2.7862 \quad a_4 = -0.23091 \quad a_5 = 0.0072923$$

$$y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072923x^5, 2 \leq x \leq 10.6$$

Fifth Order Interpolation (contd)

$$y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072923x^5, 2 \leq x \leq 10.6$$

