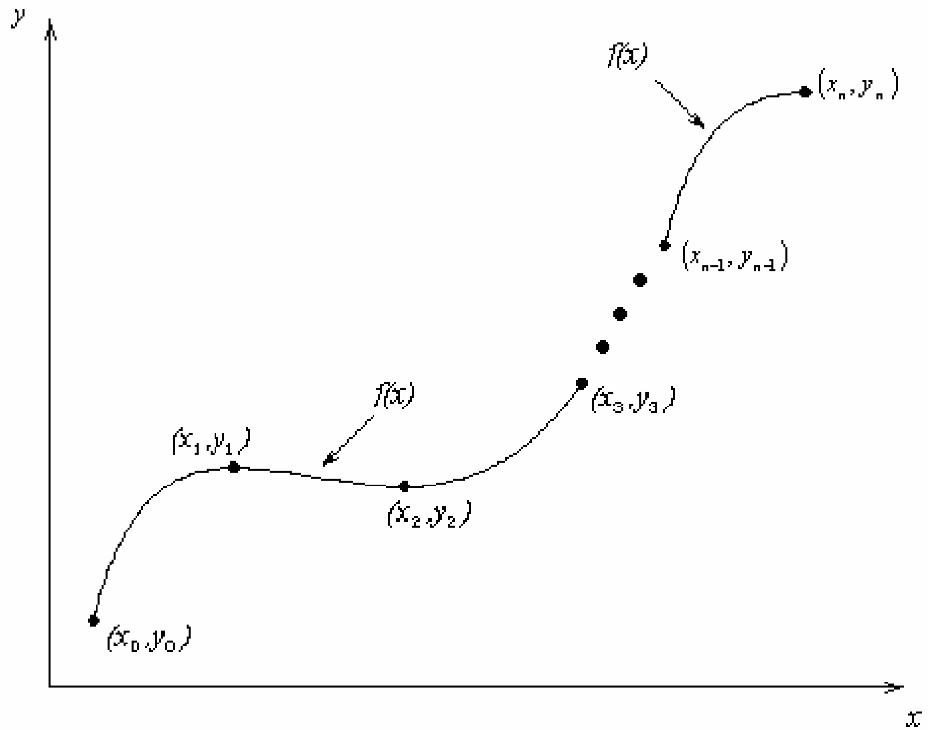
Interpolation

Topic: Lagrangian Interpolation

Major: Computer Science

What is Interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

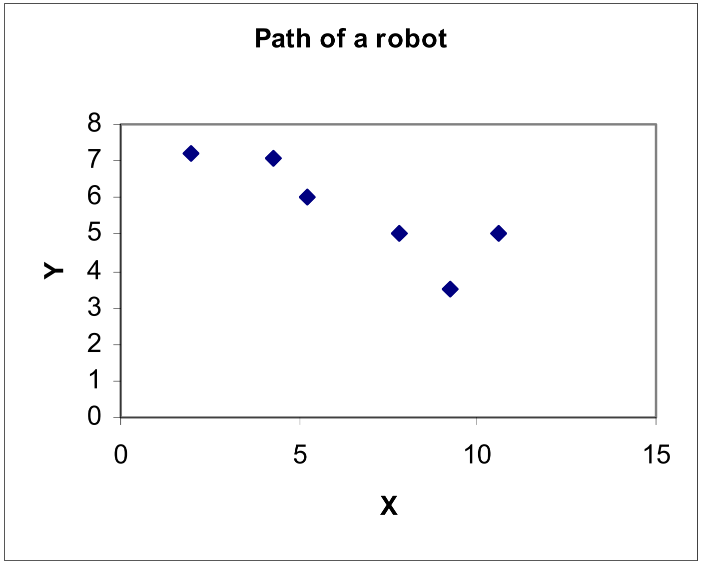
$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Example

The path of a rapid laser is given by these specifications. If the laser is traversing from x = 2 to x = 4.25 in a linear path, find the value of y at x = 4 using the Langragian method.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6
7.81	5
9.2	3.5
10.6	5



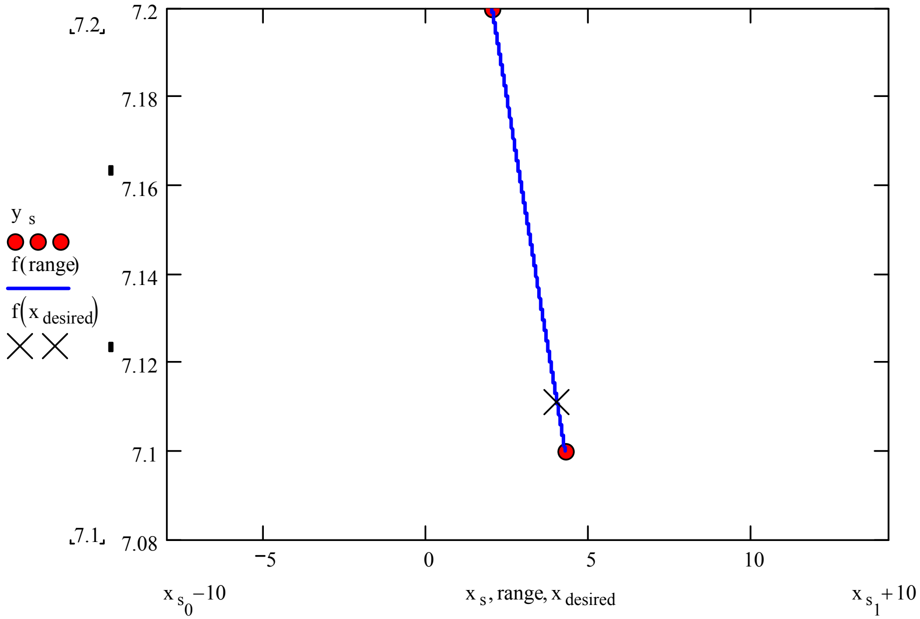
Linear Interpolation

$$y(x) = \sum_{i=0}^{1} L_i(x) y(x_i)$$

$$= L_0(x) y(x_0) + L_1(x) y(x_1)$$

$$x_0 = 2.00, y(x_0) = 7.2$$

$$x_1 = 4.25, y(x_1) = 7.1$$



Linear Interpolation (contd)

$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{x - x_j}{x_0 - x_j} = \frac{x - x_1}{x_0 - x_1}$$

$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^{1} \frac{x - x_j}{x_1 - x_j} = \frac{x - x_0}{x_1 - x_0}$$

$$y(x) = \frac{x - x_1}{x_0 - x_1} y(x_0) + \frac{x - x_0}{x_1 - x_0} y(x_1) = \frac{x - 4.25}{2.00 - 4.25}(7.2) + \frac{x - 2.00}{4.25 - 2.00}(7.1)$$

$$y(4.00) = \frac{4.00 - 4.25}{2.00 - 4.25}(7.2) + \frac{4.00 - 2.00}{4.25 - 2.00}(7.1) = 0.11111(7.2) + 0.88889(7.1)$$

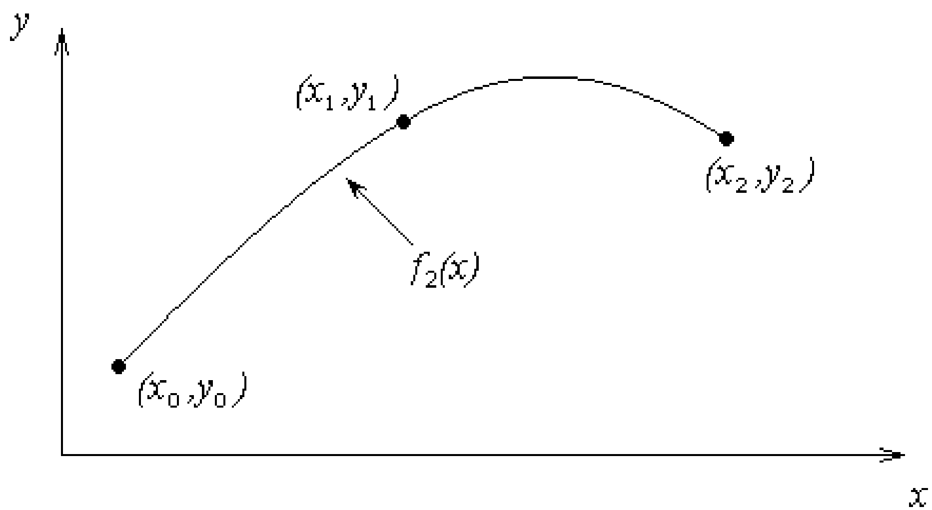
$$= 7.1111 \ in.$$

Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the value of y given by

$$y(x) = \sum_{i=0}^{2} L_i(x)\, y(x_i)$$

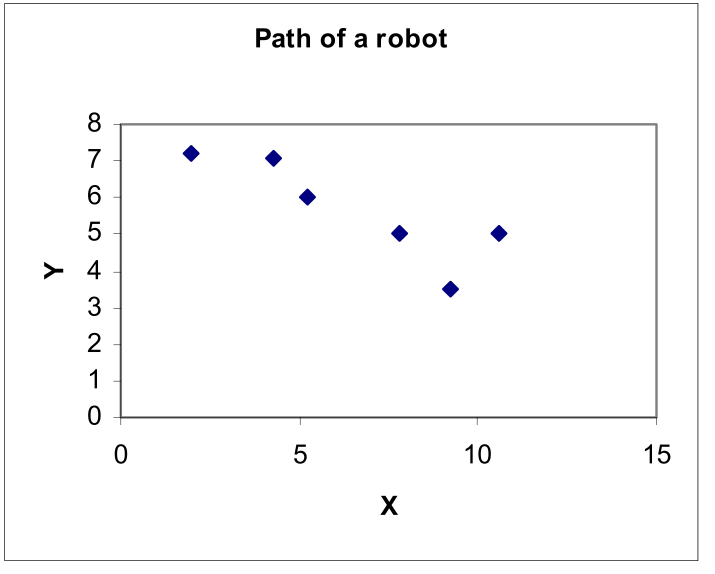
$$= L_0(x)\, y(x_0) + L_1(x)\, y(x_1) + L_2(x)\, y(x_2)$$



Example

The path of a rapid laser is given by these specifications. If the laser is traversing from x = 2 to x = 4.25 to x=5.25 using a quadratic linear path, find the value of y at x = 4 using the Lagrangian method.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6
7.81	5
9.2	3.5
10.6	5



Quadratic Interpolation

$$x_o = 2.00, \quad y(x_o) = 7.2$$

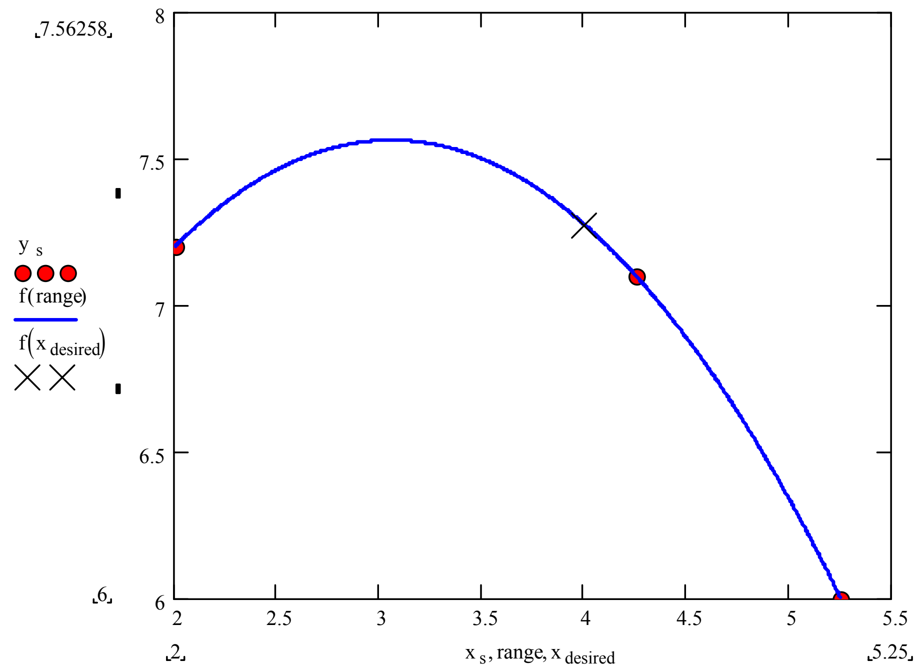
$$x_1 = 4.25, \quad y(x_1) = 7.1$$

$$x_2 = 5.25, \quad y(x_2) = 6.0$$

$$L_0(x) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{x - x_j}{x_0 - x_j} = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{x - x_j}{x_1 - x_j} = \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)$$

$$L_2(x) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{x - x_j}{x_2 - x_j} = \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)$$



Quadratic Interpolation (contd)

$$y(x)=\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)y(x_0)+\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)y(x_1)+\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)y(x_2)$$

$$y(4.00)=\frac{(4.00-4.25)(4.00-5.25)}{(2.00-4.25)(2.00-5.25)}(7.2)+\frac{(4.00-2.00)(4.00-5.25)}{(4.25-2.00)(4.25-5.25)}(7.1)+\frac{(4.00-2.00)(4.00-4.25)}{(5.25-2.00)(5.25-4.25)}(6.0)$$

$$=(0.042735)(7.2)+(1.1111)(7.1)+(-0.15385)(6.0)$$

$$=7.2735\ in.$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a|=\left|\frac{7.2735-7.1111}{7.2735}\right|\times 100$$

$$=2.2327\%$$

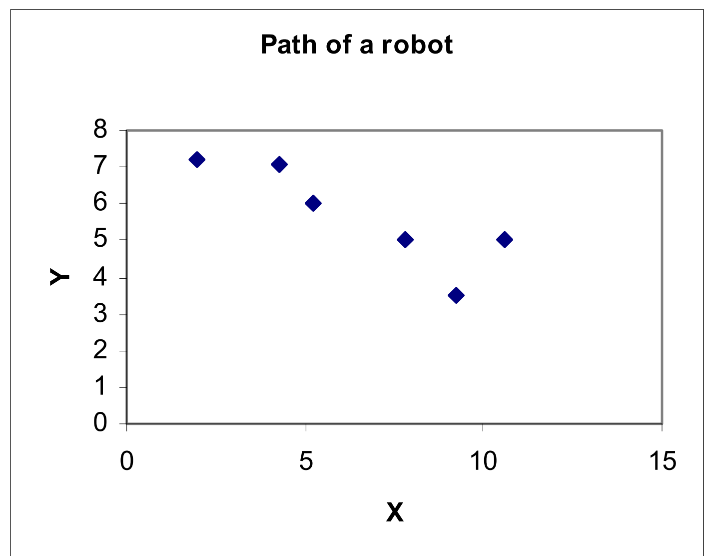
Comparison Table

Order of Polynomial	1	2
Temperature °C	35.809	35.089
Absolute Relative Approximate Error	----------	2.0544%

Example

The path of a rapid laser is given by these specifications. Find the path traversed by the laser using the Lagrange method and a fifth order polynomial.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6
7.81	5
9.2	3.5
10.6	5



Fifth Order Interpolation

$$y(x) = \sum_{i=0}^{5} L_i(x)y(x_i)$$

$$= L_0(x)y(x_0) + L_1(x)y(x_1) + L_2(x)y(x_2) + L_3(x)y(x_3) + L_4(x)y(x_4) + L_5(x)y(x_5)$$

$x_o = 2.00, \quad y(x_o) = 7.2$

$x_1 = 4.25, \quad y(x_1) = 7.1$

$x_2 = 5.25, \quad y(x_2) = 6.0$

$x_3 = 7.81, \quad y(x_3) = 5.0$

$x_4 = 9.20, \quad y(x_4) = 3.5$

$x_5 = 10.60, \quad y(x_5) = 5.0$

Fifth Order Interpolation (contd)

$$L_0(x)=\prod_{\substack{j=0\\j\neq 0}}^{5}\frac{x-x_j}{x_0-x_j}=\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)\left(\frac{x-x_3}{x_0-x_3}\right)\left(\frac{x-x_4}{x_0-x_4}\right)\left(\frac{x-x_5}{x_0-x_5}\right)$$

$$L_1(x)=\prod_{\substack{j=0\\j\neq 1}}^{5}\frac{x-x_j}{x_1-x_j}=\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)\left(\frac{x-x_4}{x_1-x_4}\right)\left(\frac{x-x_5}{x_1-x_5}\right)$$

$$L_2(x)=\prod_{\substack{j=0\\j\neq 2}}^{5}\frac{x-x_j}{x_2-x_j}=\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right)\left(\frac{x-x_4}{x_2-x_4}\right)\left(\frac{x-x_5}{x_2-x_5}\right)$$

$$L_3(x)=\prod_{\substack{j=0\\j\neq 3}}^{5}\frac{x-x_j}{x_3-x_j}=\left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)\left(\frac{x-x_4}{x_3-x_4}\right)\left(\frac{x-x_5}{x_3-x_5}\right)$$

$$L_4(x)=\prod_{\substack{j=0\\j\neq 4}}^{5}\frac{x-x_j}{x_4-x_j}=\left(\frac{x-x_0}{x_4-x_0}\right)\left(\frac{x-x_1}{x_4-x_1}\right)\left(\frac{x-x_2}{x_4-x_2}\right)\left(\frac{x-x_3}{x_4-x_3}\right)\left(\frac{x-x_5}{x_4-x_5}\right)$$

$$L_5(x)=\prod_{\substack{j=0\\j\neq 5}}^{5}\frac{x-x_j}{x_5-x_j}=\left(\frac{x-x_0}{x_5-x_0}\right)\left(\frac{x-x_1}{x_5-x_1}\right)\left(\frac{x-x_2}{x_5-x_2}\right)\left(\frac{x-x_3}{x_5-x_3}\right)\left(\frac{x-x_4}{x_5-x_4}\right)$$

Fifth Order Polynomial (contd)

$$
\begin{aligned}
y(x) = {} & \left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)\left(\frac{x-x_3}{x_0-x_3}\right)\left(\frac{x-x_4}{x_0-x_4}\right)\left(\frac{x-x_5}{x_0-x_5}\right)\left(\frac{x-x_6}{x_0-x_6}\right)y(x_0) \\
& + \left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)\left(\frac{x-x_4}{x_1-x_4}\right)\left(\frac{x-x_5}{x_1-x_5}\right)\left(\frac{x-x_6}{x_1-x_6}\right)y(x_1) \\
& + \left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right)\left(\frac{x-x_4}{x_2-x_4}\right)\left(\frac{x-x_5}{x_2-x_5}\right)\left(\frac{x-x_6}{x_2-x_6}\right)y(x_2) \\
& + \left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)\left(\frac{x-x_4}{x_3-x_4}\right)\left(\frac{x-x_5}{x_3-x_5}\right)\left(\frac{x-x_6}{x_3-x_6}\right)y(x_3) \\
& + \left(\frac{x-x_0}{x_4-x_0}\right)\left(\frac{x-x_1}{x_4-x_1}\right)\left(\frac{x-x_2}{x_4-x_2}\right)\left(\frac{x-x_3}{x_4-x_3}\right)\left(\frac{x-x_5}{x_4-x_5}\right)\left(\frac{x-x_6}{x_4-x_6}\right)y(x_4) \\
& + \left(\frac{x-x_0}{x_5-x_0}\right)\left(\frac{x-x_1}{x_5-x_1}\right)\left(\frac{x-x_2}{x_5-x_2}\right)\left(\frac{x-x_3}{x_5-x_3}\right)\left(\frac{x-x_4}{x_5-x_4}\right)\left(\frac{x-x_6}{x_5-x_6}\right)y(x_5)
\end{aligned}
$$

Fifth Order Polynomial (contd)

$$
\begin{aligned}
y(x) &= \frac{(x-4.25)(x-5.25)(x-7.81)(x-9.20)(x-10.60)}{(2.00-4.25)(2.00-5.25)(2.00-7.81)(2.00-9.20)(2.00-10.60)}(7.2) \\
&+ \frac{(x-2.00)(x-5.25)(x-7.81)(x-9.20)(x-10.60)}{(4.25-2.00)(4.25-5.25)(4.25-7.81)(4.25-9.20)(4.25-10.60)}(7.1) \\
&+ \frac{(x-2.00)(x-4.25)(x-7.81)(x-9.20)(x-10.60)}{(5.25-2.00)(5.25-4.25)(5.25-7.81)(5.25-9.20)(5.25-10.60)}(6.0) \\
&+ \frac{(x-2.00)(x-4.25)(x-5.25)(x-9.20)(x-10.60)}{(7.81-2.00)(7.81-4.25)(7.81-5.25)(7.81-9.20)(7.81-10.60)}(5.0) \\
&+ \frac{(x-2.00)(x-4.25)(x-5.25)(x-7.81)(x-10.60)}{(9.20-2.00)(9.20-4.25)(9.20-5.25)(9.20-7.81)(9.20-10.60)}(3.5) \\
&+ \frac{(x-2.00)(x-4.25)(x-5.25)(x-7.81)(x-9.20)}{(10.60-2.00)(10.60-4.25)(10.60-5.25)(10.60-7.81)(10.60-9.20)}(5.0)
\end{aligned}
$$

Fifth Order Polynomial (contd)

$$= \frac{x^5 - 37.11x^4 + 536.77x^3 - 3773.2x^2 + 12862x - 16994}{-365.38}$$

$$+ \frac{x^5 - 34.86x^4 + 462.83x^3 - 2879.7x^2 + 8169.5x - 7997.1}{35.461}$$

$$+ \frac{x^5 - 33.86x^4 + 433.22x^3 - 2572.3x^2 + 6903.5x - 6473.9}{-29.304}$$

$$+ \frac{x^5 - 31.3x^4 + 366.53x^3 - 1984.1x^2 + 4912.4x - 4351.8}{41.069}$$

$$+ \frac{x^5 - 29.91x^4 + 335.81x^3 - 1757.2x^2 + 4241.6x - 3694.3}{-78.273}$$

$$+ \frac{x^5 - 28.51x^4 + 308.78x^3 - 1573.7x^2 + 3727.5x - 3206.4}{228.24}$$

$$y(x) = -30.900 + 41.351x - 15.858x^2 + 2.7865x^3$$

$$-0.23093x^4 + 0.0072929x^5, \quad 2 \le x \le 10.6$$

Fifth Order Polynomial (contd)

$$y(x) = -30.900 + 41.351x - 15.858x^2 + 2.7865x^3$$
$$-0.23093x^4 + 0.0072929x^5, \quad 2 \le x \le 10.6$$