

Interpolation



Topic: Newton's Divided
Difference Polynomial Method

Major: Computer Science



What is Interpolation ?

Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, find the value of 'y' at a value of 'x' that is not given.





Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Newton's Divided Difference Method

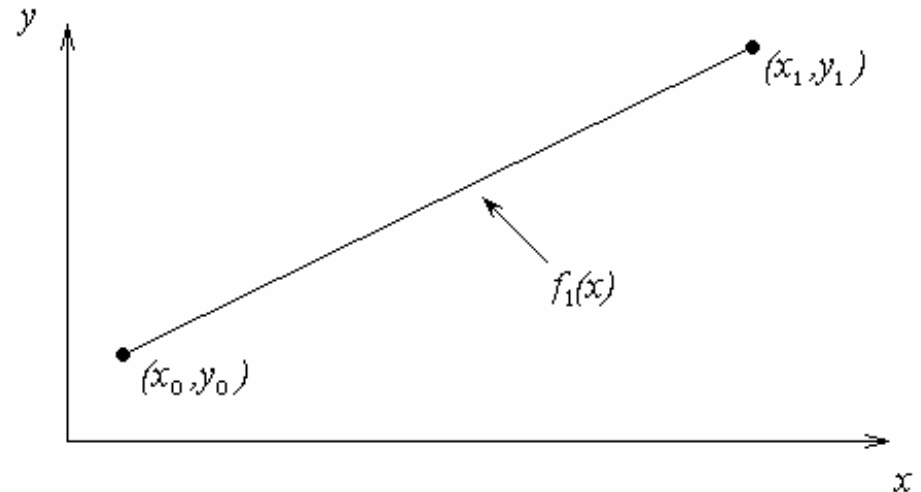
Linear interpolation: Given (x_0, y_0) , (x_1, y_1) , pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$

where

$$b_0 = f(x_0)$$

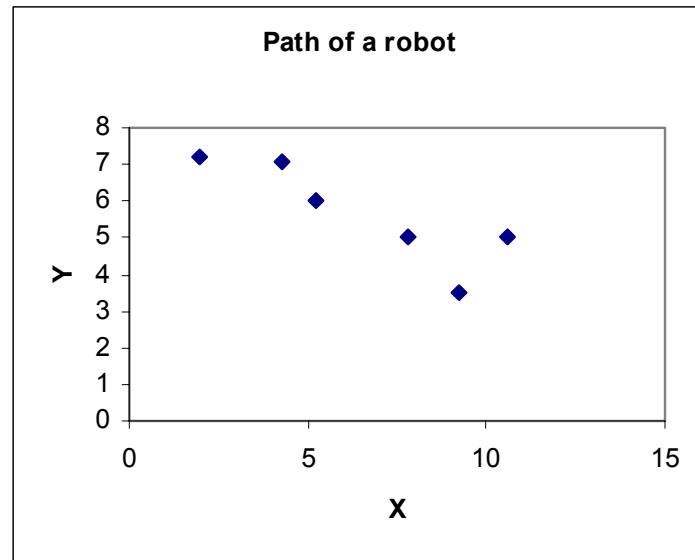
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



Example

The path of a rapid laser is given by these specifications. If the laser is traversing from $x = 2$ to $x = 4.25$ in a linear path, find the value of y at $x = 4$ using the Newton's Divided difference polynomial method.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6
7.81	5
9.2	3.5
10.6	5



Linear Interpolation

$$y(x) = b_0 + b_1(x - x_0)$$

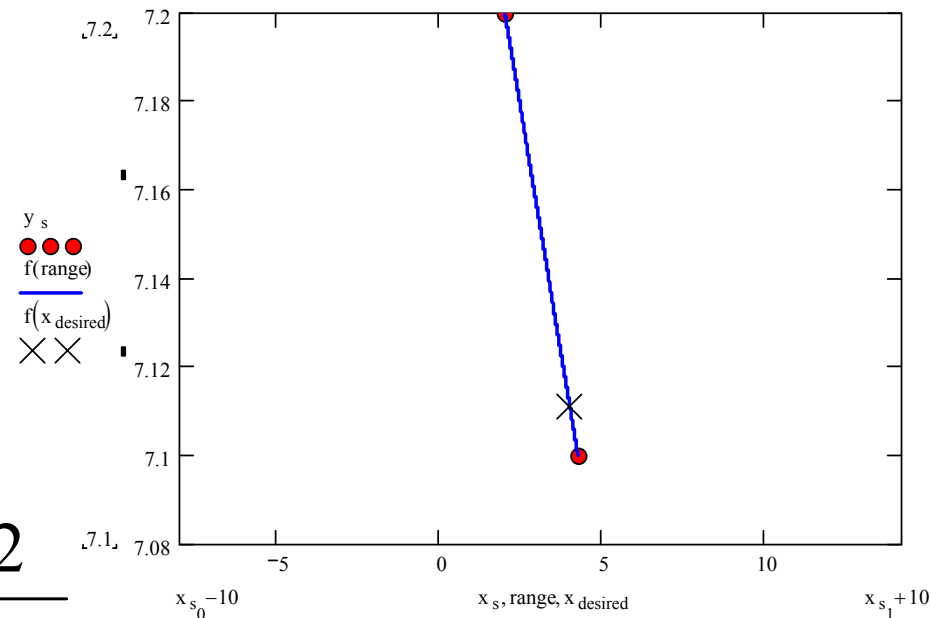
$$x_0 = 2.00, y(x_0) = 7.2$$

$$x_1 = 4.25, y(x_1) = 7.1$$

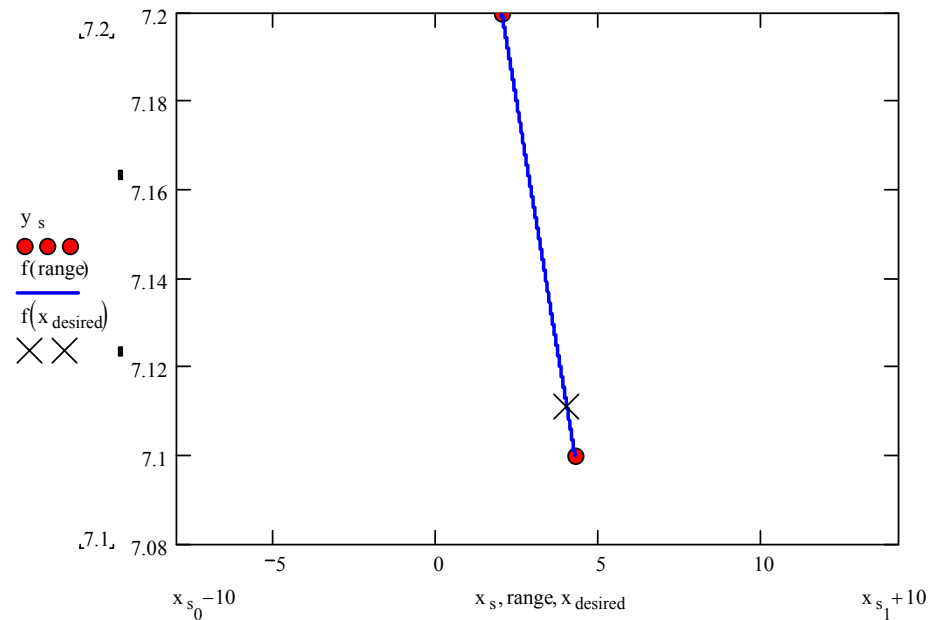
$$b_0 = y(x_0) = 7.2$$

$$b_1 = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{7.1 - 7.2}{4.25 - 2.00}$$

$$= -0.04444$$



Linear Interpolation (contd)



$$\begin{aligned}y(x) &= b_0 + b_1(x - x_0) \\ &= 7.2 - 0.04444(x - 2.00), \quad 2.00 \leq x \leq 4.25\end{aligned}$$

At $x = 4$

$$\begin{aligned}x(4.00) &= 7.2 - 0.04444(4.00 - 2.00) \\ &= 7.1111 \text{ in.}\end{aligned}$$



Quadratic Interpolation

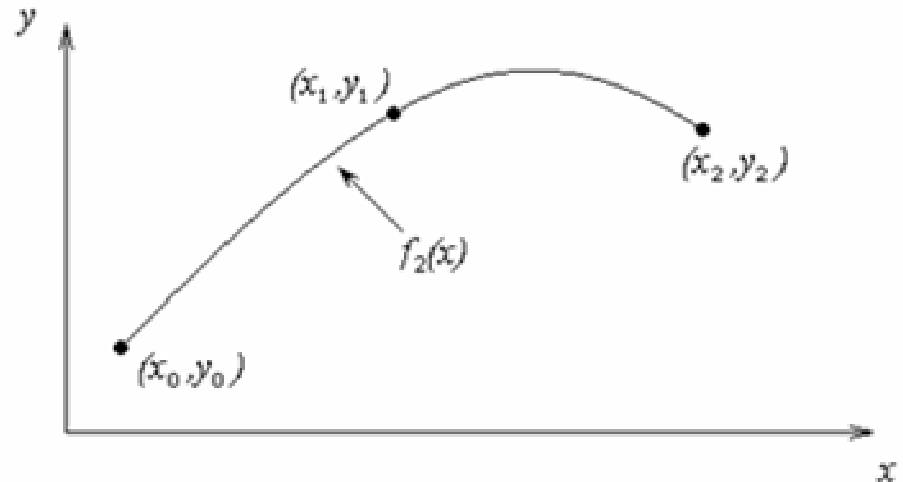
Given (x_0, y_0) , (x_1, y_1) , and (x_2, y_2) , fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

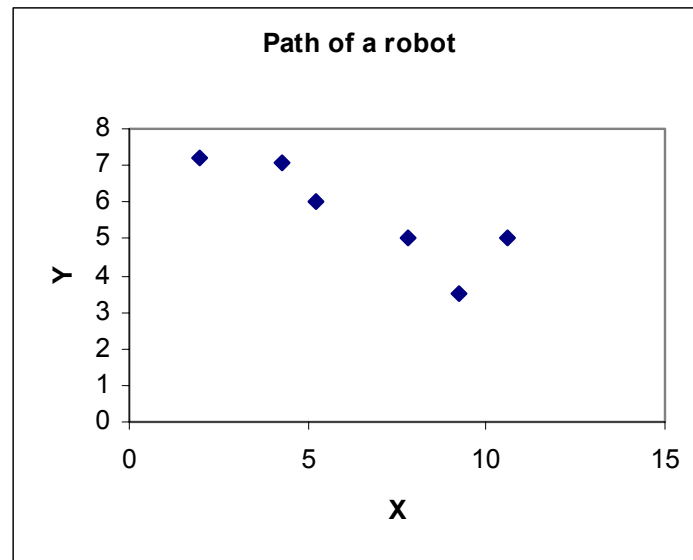


Example

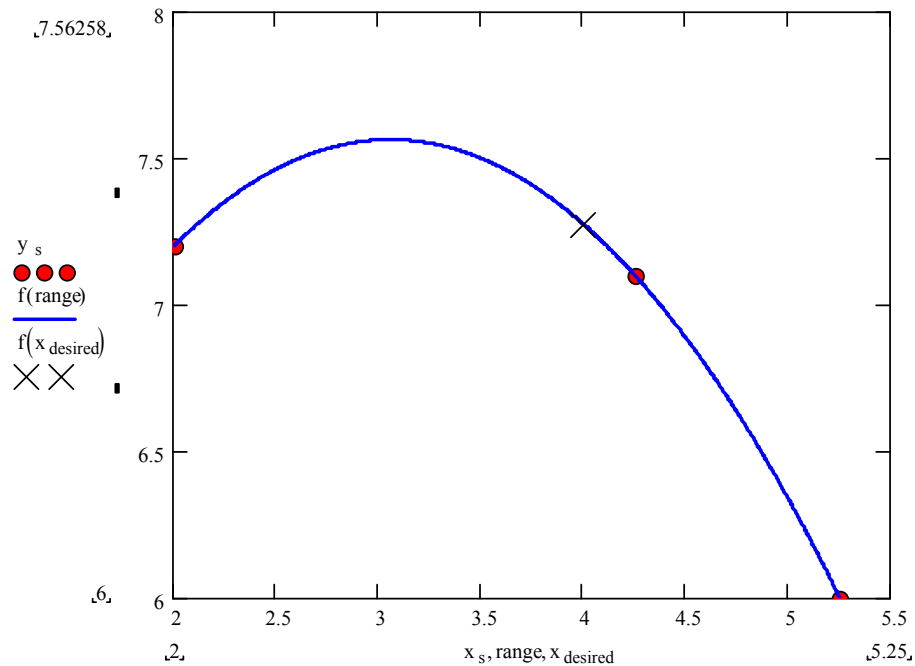
The path of a rapid laser is given by these specifications.

If the laser is traversing from $x = 2$ to $x = 4.25$ to $x = 5.25$ using a quadratic path, find the value of y at $x = 4$ using the Newton's Divided difference polynomial method.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6
7.81	5
9.2	3.5
10.6	5



Quadratic Interpolation (contd)



$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$x_0 = 2.00, \quad y(x_0) = 7.2$$

$$x_1 = 4.25, \quad y(x_1) = 7.1$$

$$x_2 = 5.25, \quad y(x_2) = 6.0$$



Quadratic Interpolation (contd)

$$b_0 = y(x_0)$$

$$= 7.2$$

$$b_1 = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{7.1 - 7.2}{4.25 - 2.00}$$

$$= -0.04444$$

$$b_2 = \frac{\frac{y(x_2) - y(x_1)}{x_2 - x_1} - \frac{y(x_1) - y(x_0)}{x_1 - x_0}}{x_2 - x_0} = \frac{\frac{6.0 - 7.1}{5.25 - 4.25} - \frac{7.1 - 7.2}{4.25 - 2.00}}{5.25 - 2.00}$$

$$= \frac{-1.1 + 0.04444}{3.25}$$

$$= -0.32479$$



Quadratic Interpolation (contd)

$$\begin{aligned}y(x) &= b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \\ &= 7.2 - 0.04444(x - 2.00) - 0.32479(x - 2.00)(x - 4.25), \quad 2.00 \leq x \leq 5.25\end{aligned}$$

At $x = 4$,

$$\begin{aligned}y(4.00) &= 7.2 - 0.04444(4.00 - 2.00) - 0.32479(4.00 - 2.00)(4.00 - 4.25) \\ &= 7.2735 \text{ in.}\end{aligned}$$

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the first and second order polynomial is

$$\begin{aligned}|\epsilon_a| &= \left| \frac{7.2735 - 7.1111}{7.2735} \right| \times 100 \\ &= 2.2327\%\end{aligned}$$



General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$



General Form

Given $(n + 1)$ data points, $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + \dots + b_n(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

\vdots

$$b_{n-1} = f[x_{n-1}, x_{n-2}, \dots, x_0]$$

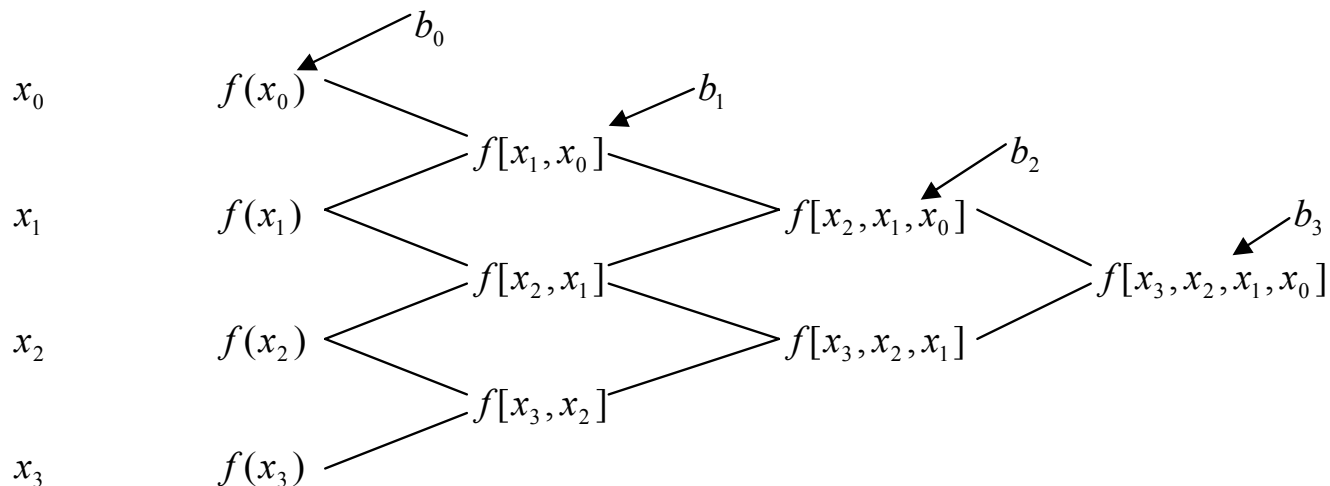
$$b_n = f[x_n, x_{n-1}, \dots, x_0]$$



General form

The third order polynomial, given (x_0, y_0) , (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) , is

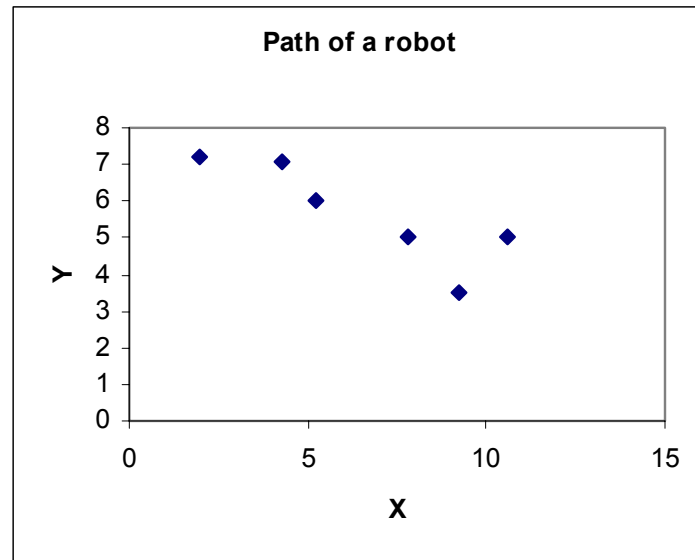
$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$



Example

The path of a rapid laser is given by these specifications. Find the path traversed by the laser using the Newton's Divided Difference Polynomial method.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6
7.81	5
9.2	3.5
10.6	5





Example

The value of y profile is chosen as

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2) \\ + b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) + b_5(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

Using the six points,

$x_0 = 2.00,$	$y(x_0) = 7.2$	$x_1 = 4.25,$	$y(x_1) = 7.1$
$x_2 = 5.25,$	$y(x_2) = 6.0$	$x_3 = 7.81,$	$y(x_3) = 5.0$
$x_4 = 9.20,$	$y(x_4) = 3.5$	$x_5 = 10.60,$	$y(x_5) = 5.0$

The values of the constants are found to be

$b_0 = 7.2$	$b_1 = -0.04444$	$b_2 = -0.32479$
$b_3 = 0.090198$	$b_4 = -0.023009$	$b_5 = 0.0072922$



Example

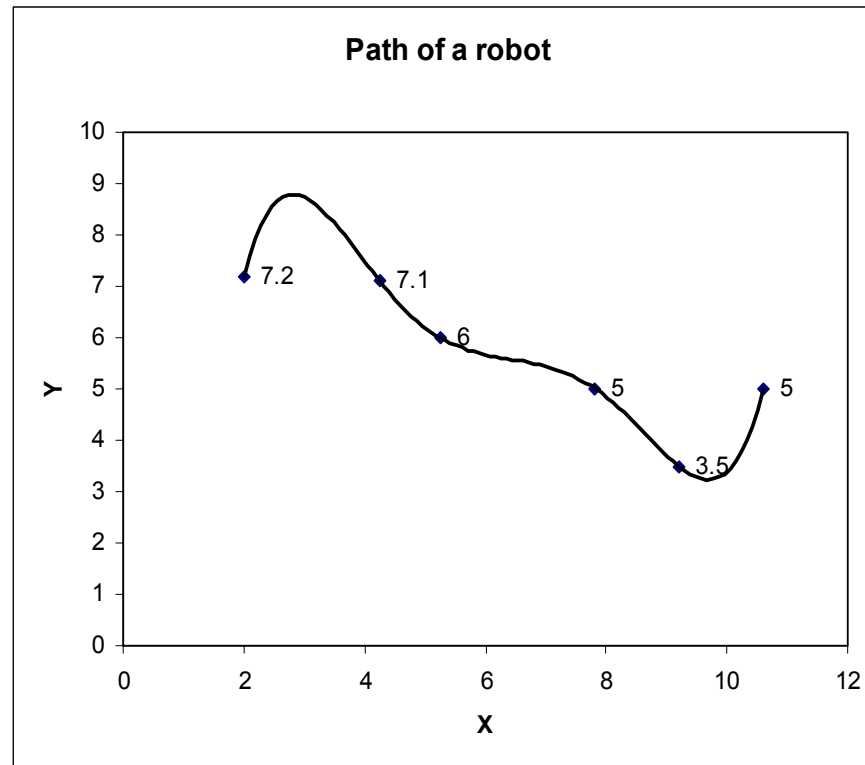
Hence

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2) \\ + b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) + b_5(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

$$= 7.2 - 0.04444(x - 2) - 0.32479(x - 2)(x - 4.25) \\ + 0.090198(x - 2)(x - 4.25)(x - 5.25) \\ - 0.023009(x - 2)(x - 4.25)(x - 5.25)(x - 7.81) \\ + 0.0072922(x - 2)(x - 4.25)(x - 5.25)(x - 7.81)(x - 9.2)$$

$$y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072922x^5, \quad 2 \leq x \leq 10.6$$

Example



$$y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072922x^5, \quad 2 \leq x \leq 10.6$$