

# Interpolation

Topic: Spline Interpolation Method

Major: Computer Science

# What is Interpolation ?

Given  $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ , find the value of 'y' at a value of 'x' that is not given.





# Interpolants

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Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

# Why Splines ?

$$f(x) = \frac{1}{1 + 25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

$x$	$y = \frac{1}{1 + 25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

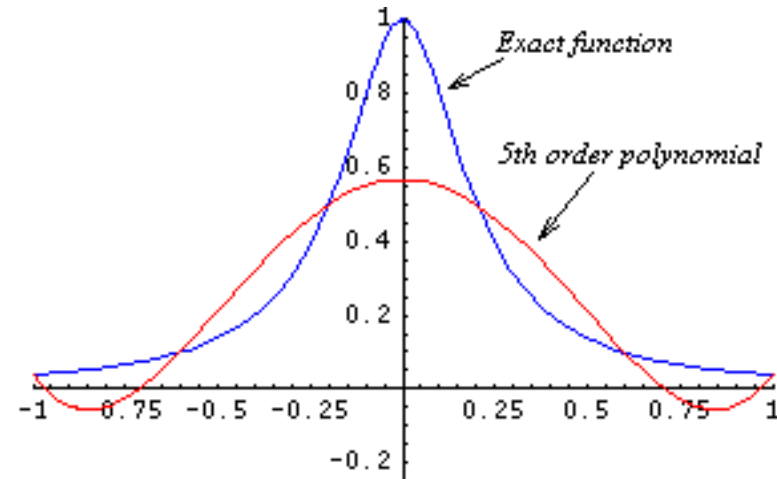
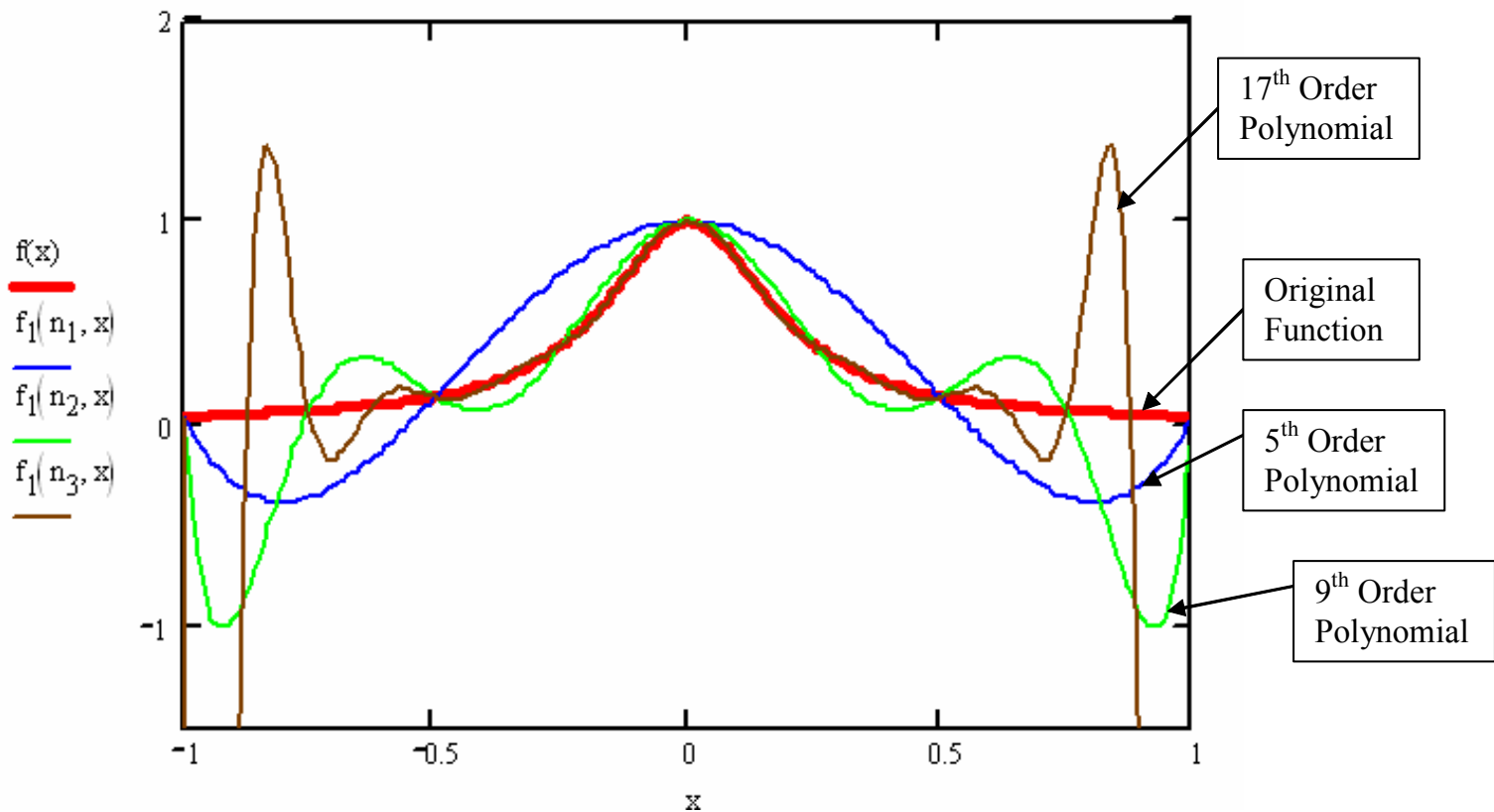


Figure : 5<sup>th</sup> order polynomial vs. exact function

# Why Splines ?



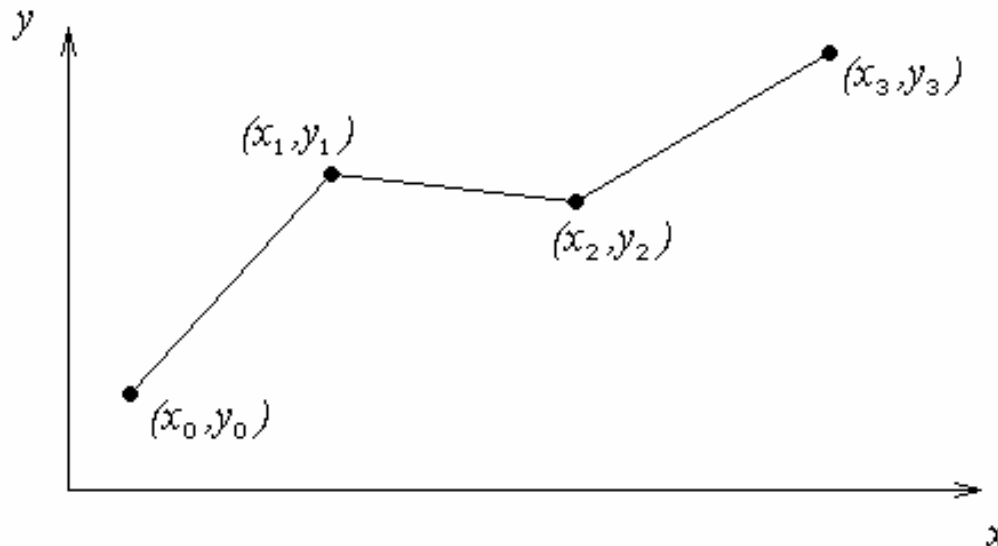
**Figure : Higher order polynomial interpolation is a bad idea**



# Linear Interpolation

Given  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ , fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by  $(y_i = f(x_i))$

**Figure : Linear splines**





# Linear Interpolation (contd)

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$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \quad x_0 \leq x \leq x_1$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \quad x_1 \leq x \leq x_2$$

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$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \leq x \leq x_n$$

Note the terms of

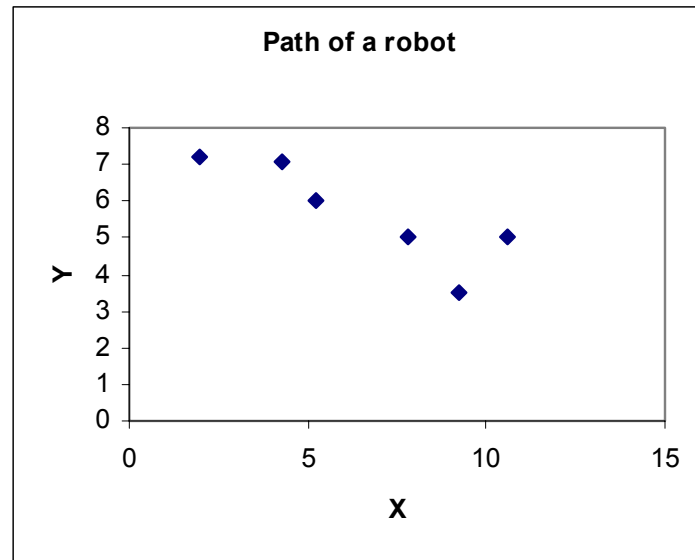
$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between  $x_{i-1}$  and  $x_i$ .

# Example

The path of a rapid laser is given by these specifications. If the laser is traversing from  $x = 2$  to  $x = 4.25$  in a linear path, find the value of  $y$  at  $x = 4$  using the linear spline interpolation method.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6
7.81	5
9.2	3.5
10.6	5



# Linear Interpolation

$$x_0 = 2.00, \quad y(x_0) = 7.2$$

$$x_1 = 4.25, \quad y(x_1) = 7.1$$

$$y(x) = y(x_0) + \frac{y(x_1) - y(x_0)}{x_1 - x_0} (x - x_0)$$

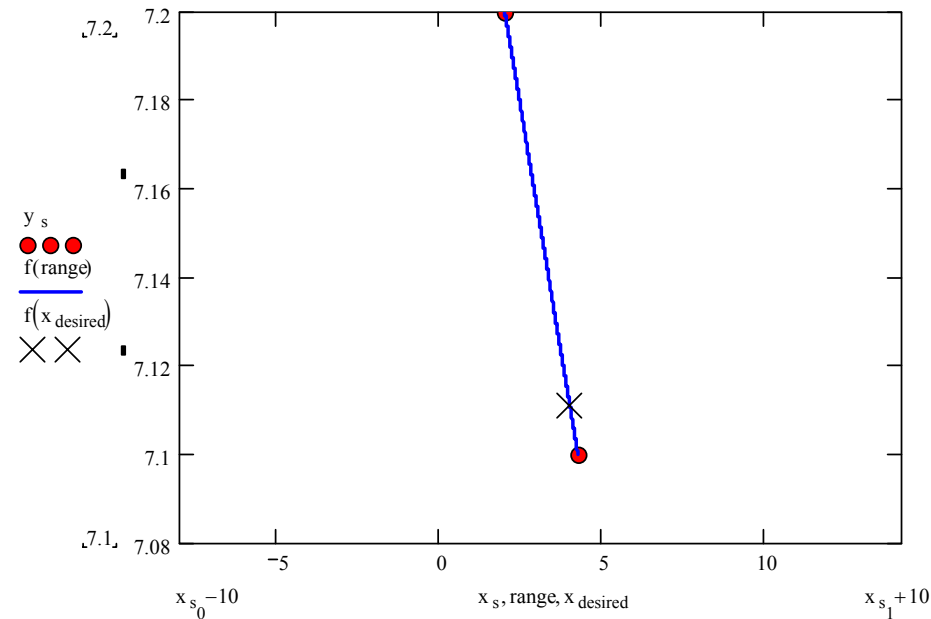
$$= 7.2 + \frac{7.1 - 7.2}{4.25 - 2.00} (x - 2.00)$$

$$y(x) = 7.2 - 0.04444(x - 2.00)$$

At  $x = 4$ ,

$$y(4.00) = 7.2 - 0.04444(4.00 - 2.00)$$

$$= 7.1111 \text{ in.}$$





# Linear Interpolation (contd)

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Find the path of the robot if it follows linear splines.

The linear spline connecting  $x = 2.00$  and  $x = 4.25$ .

$$y(x) = 7.2 - 0.04444(x - 2.00), \quad 2.00 \leq x \leq 4.25$$

Similarly

$$y(x) = 7.1 - 1.1(x - 4.25), \quad 4.25 \leq x \leq 5.25$$

$$y(x) = 6.0 - 0.390625(x - 5.25), \quad 5.25 \leq x \leq 7.81$$

$$y(x) = 5.0 - 1.07914(x - 7.81), \quad 7.81 \leq x \leq 9.20$$

$$y(x) = 3.5 + 1.07143(x - 9.20), \quad 9.20 \leq x \leq 10.60$$



# Linear Interpolation (contd)

Find the length of the path traversed by the robot following linear splines.

The length of the robot's path can be found by simply adding the length of the line segments together. The length of a straight line from one point

$(x_0, y_0)$  to another point  $(x_1, y_1)$  is given by  $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$ .

Hence, the length of the linear splines from  $x = 2.00$  to  $x = 10.60$  is

$$\begin{aligned} L = & \sqrt{(4.25 - 2.00)^2 + (7.1 - 7.2)^2} + \sqrt{(5.25 - 4.25)^2 + (6.0 - 7.1)^2} \\ & + \sqrt{(7.81 - 5.25)^2 + (5.0 - 6.0)^2} + \sqrt{(9.20 - 7.81)^2 + (3.5 - 5.0)^2} \\ & + \sqrt{(10.60 - 9.20)^2 + (5.0 - 3.5)^2} \end{aligned}$$

$$L=10.584 \text{ m}$$

# Quadratic Interpolation

Given  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ , fit quadratic splines through the data. The splines are given by

$$f(x) = a_1x^2 + b_1x + c_1, \quad x_0 \leq x \leq x_1$$

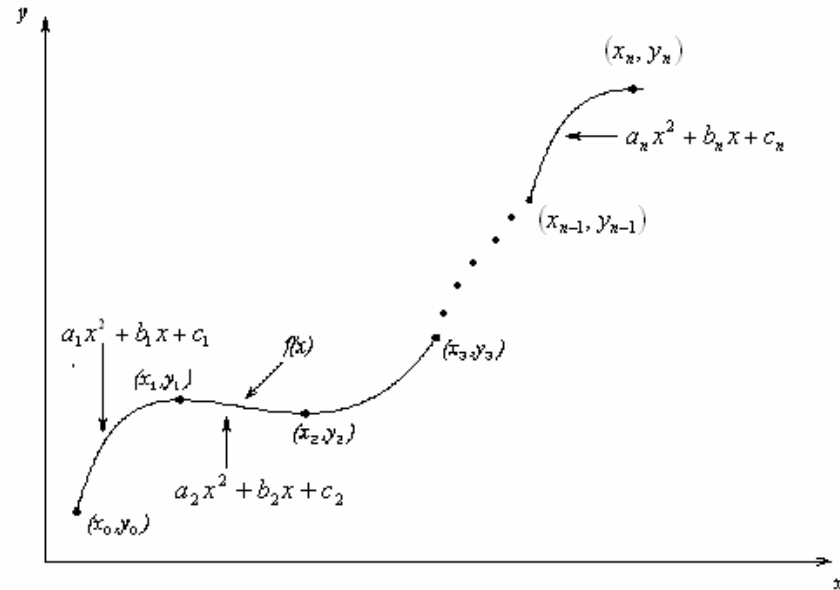
$$= a_2x^2 + b_2x + c_2, \quad x_1 \leq x \leq x_2$$

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$$= a_nx^2 + b_nx + c_n, \quad x_{n-1} \leq x \leq x_n$$

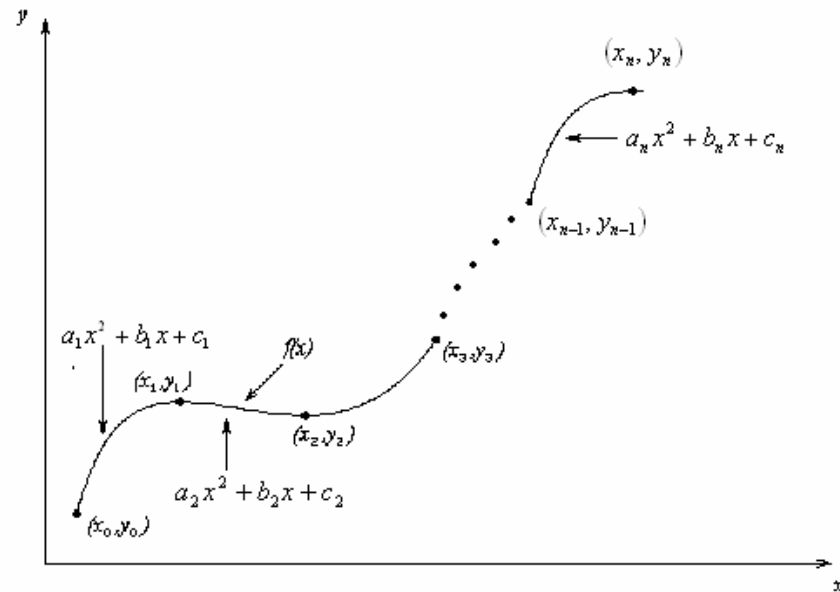


Find  $a_i, b_i, c_i, i = 1, 2, \dots, n$

# Quadratic Interpolation (contd)

Each quadratic spline goes through two consecutive data points

$$\begin{aligned}
 a_1 x_0^2 + b_1 x_0 + c_1 &= f(x_0) \\
 a_1 x_1^2 + b_1 x_1 + c_1 &= f(x_1) \\
 &\vdots \\
 a_i x_{i-1}^2 + b_i x_{i-1} + c_i &= f(x_{i-1}) \\
 a_i x_i^2 + b_i x_i + c_i &= f(x_i) \\
 &\vdots \\
 a_n x_{n-1}^2 + b_n x_{n-1} + c_n &= f(x_{n-1}) \\
 a_n x_n^2 + b_n x_n + c_n &= f(x_n)
 \end{aligned}$$



This condition gives  $2n$  equations

# Quadratic Splines (contd)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1 \text{ is } 2a_1x + b_1$$

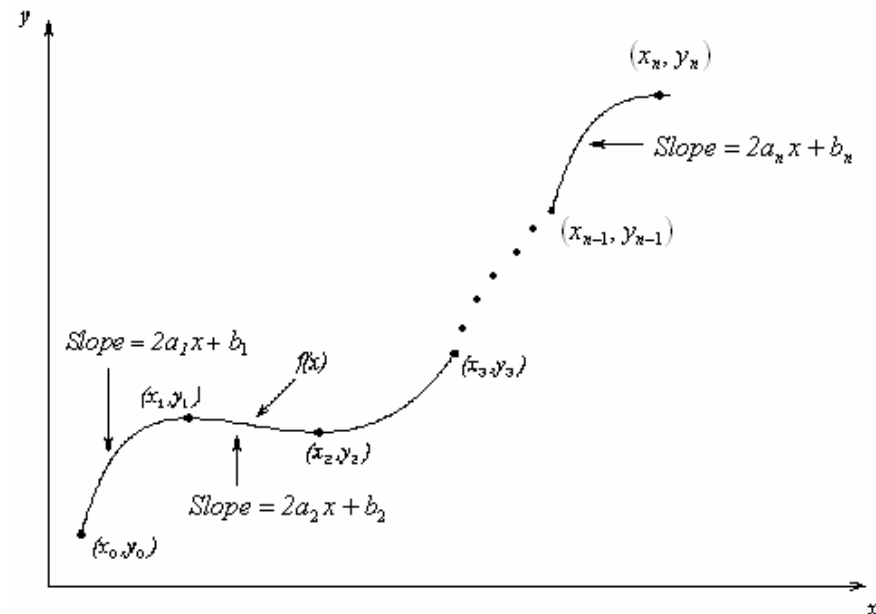
The derivative of the second spline

$$a_2x^2 + b_2x + c_2 \text{ is } 2a_2x + b_2$$

and the two are equal at  $x = x_1$  giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



# Quadratic Splines (contd)

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

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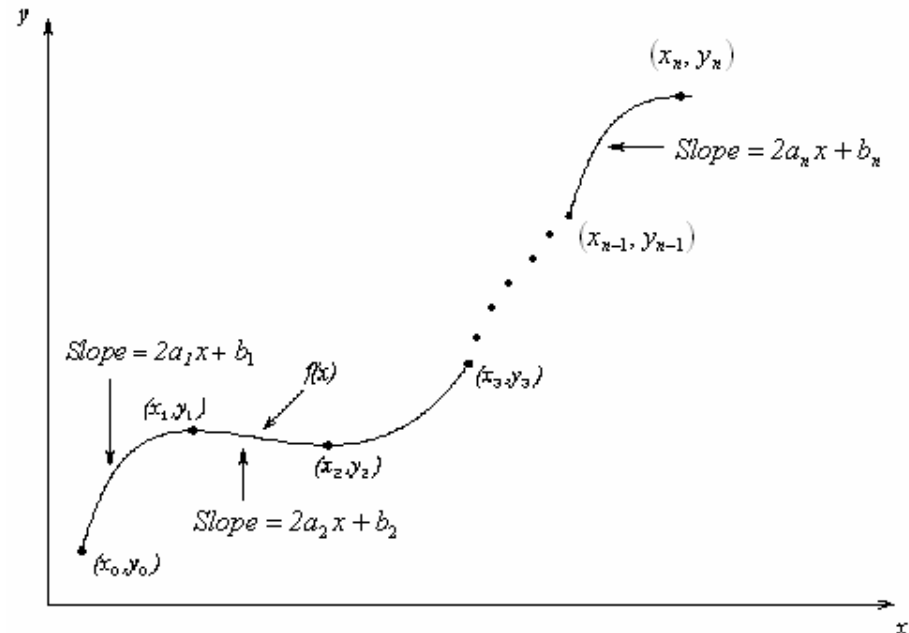
$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

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$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$



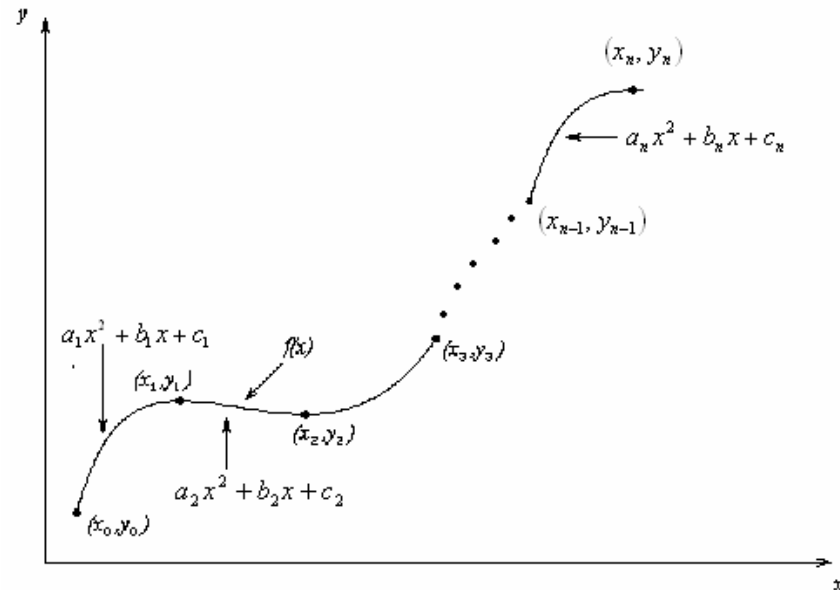
We have (n-1) such equations. The total number of equations is  $(2n) + (n - 1) = (3n - 1)$ .

We can assume that the first spline is linear, that is  $a_1 = 0$

# Quadratic Splines (contd)

This gives us '3n' equations and '3n' unknowns. Once we find the '3n' constants, we can find the function at any value of 'x' using the splines,

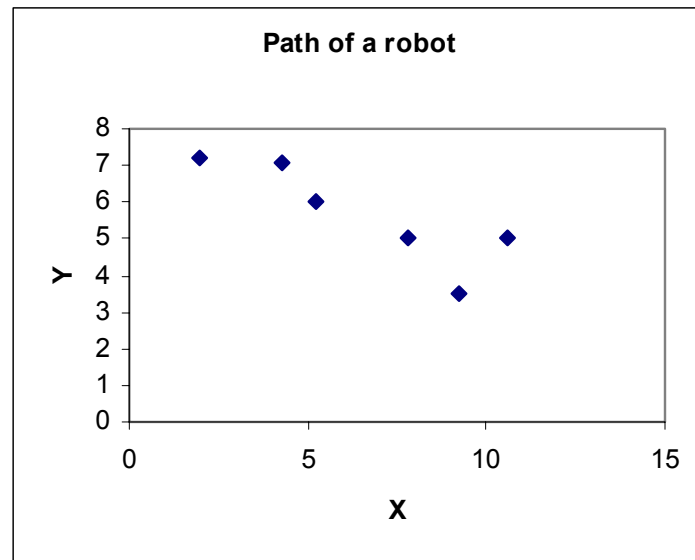
$$\begin{aligned} f(x) &= a_1x^2 + b_1x + c_1, & x_0 \leq x \leq x_1 \\ &= a_2x^2 + b_2x + c_2, & x_1 \leq x \leq x_2 \\ &\cdot \\ &\cdot \\ &= a_nx^2 + b_nx + c_n, & x_{n-1} \leq x \leq x_n \end{aligned}$$



# Example

The path of a rapid laser is given by these specifications in Cartesian co-ordinates. Find the length of the path traversed by the robot using quadratic splines.

x (m)	y (m)
2	7.2
4.25	7.1
5.25	6
7.81	5
9.2	3.5
10.6	5



# Solution

Since there are six data points,  
five quadratic splines pass through them.

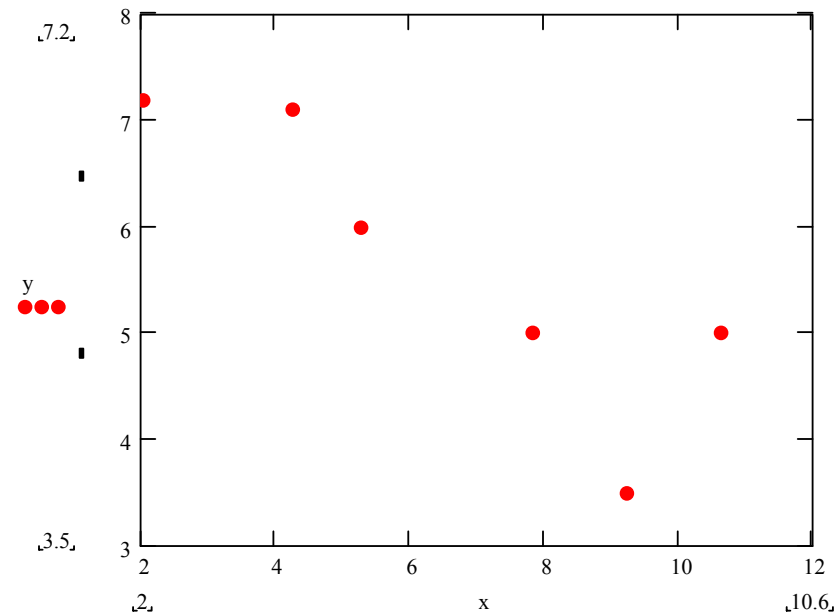
$$y(x) = a_1x^2 + b_1x + c_1, \quad 2.00 \leq x \leq 4.25$$

$$= a_2x^2 + b_2x + c_2, \quad 4.25 \leq x \leq 5.25$$

$$= a_3x^2 + b_3x + c_3, \quad 5.25 \leq x \leq 7.81$$

$$= a_4x^2 + b_4x + c_4, \quad 7.81 \leq x \leq 9.20$$

$$= a_5x^2 + b_5x + c_5, \quad 9.20 \leq x \leq 10.60$$





# Solution (contd)

Setting up the equations

Each quadratic spline passes through two consecutive data points giving

$a_1x^2 + b_1x + c_1$  passes through  $x = 2.00$  and  $x = 4.25$ ,

$$a_1(2.00)^2 + b_1(2.00) + c_1 = 7.2 \quad (1)$$

$$a_1(4.25)^2 + b_1(4.25) + c_1 = 7.1 \quad (2)$$

Similarly,

$$a_2(4.25)^2 + b_2(4.25) + c_2 = 7.1 \quad (3)$$

$$a_2(5.25)^2 + b_2(5.25) + c_2 = 6.0 \quad (4)$$

$$a_3(5.25)^2 + b_3(5.25) + c_3 = 6.0 \quad (5)$$

$$a_3(7.81)^2 + b_3(7.81) + c_3 = 5.0 \quad (6)$$

$$a_4(7.81)^2 + b_4(7.81) + c_4 = 5.0 \quad (7)$$

$$a_4(9.20)^2 + b_4(9.20) + c_4 = 3.5 \quad (8)$$

$$a_5(9.20)^2 + b_5(9.20) + c_5 = 3.5 \quad (9)$$

$$a_5(10.60)^2 + b_5(10.60) + c_5 = 5.0 \quad (10)$$

# Solution (contd)

Quadratic splines have continuous derivatives at the interior data points

At  $x = 4.25$

$$2a_1(4.25) + b_1 - 2a_2(4.25) - b_2 = 0 \quad (11)$$

At  $x = 5.25$

$$2a_2(5.25) + b_2 - 2a_3(5.25) - b_3 = 0 \quad (12)$$

At  $x = 7.81$

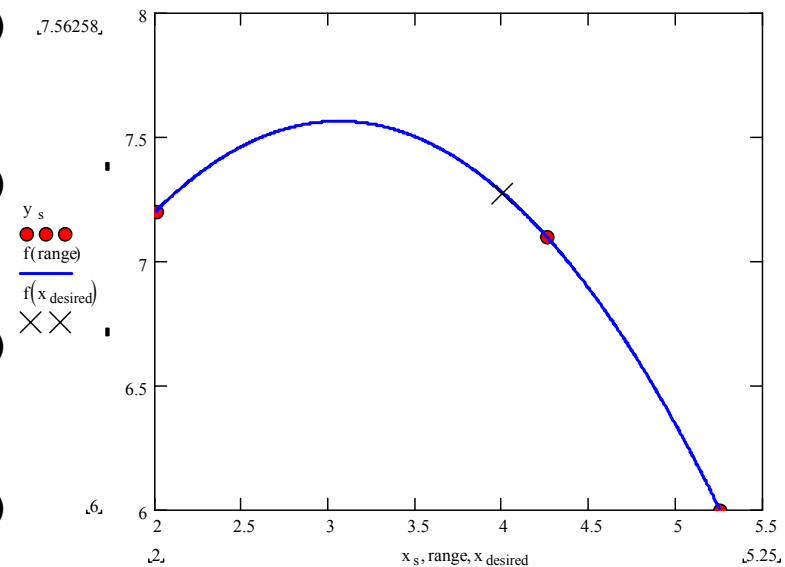
$$2a_3(7.81) + b_3 - 2a_4(7.81) - b_4 = 0 \quad (13)$$

At  $x = 9.20$

$$2a_4(9.20) + b_4 - 2a_5(9.20) - b_5 = 0 \quad (14)$$

Assuming the first spline  $a_1x^2 + b_1x + c_1$  is linear,

$$a_1 = 0 \quad (15)$$



# Solution (contd)

$$\begin{bmatrix}
 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 18.063 & 4.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 18.063 & 4.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 27.563 & 5.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 27.563 & 5.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 60.996 & 7.81 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 60.996 & 7.81 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 84.64 & 9.2 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 84.64 & 9.2 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 112.36 & 10.6 & 1 \\
 8.5 & 1 & 0 & -8.5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 10.5 & 1 & 0 & -10.5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 15.62 & 1 & 0 & -15.62 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18.4 & 1 & 0 & -18.4 & -1 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}
 \begin{bmatrix}
 a_1 \\
 b_1 \\
 c_1 \\
 a_2 \\
 b_2 \\
 c_2 \\
 a_3 \\
 b_3 \\
 c_3 \\
 a_4 \\
 b_4 \\
 c_4 \\
 a_5 \\
 b_5 \\
 c_5
 \end{bmatrix}
 =
 \begin{bmatrix}
 7.2 \\
 7.1 \\
 7.1 \\
 6.0 \\
 6.0 \\
 5.0 \\
 5.0 \\
 3.5 \\
 3.5 \\
 5.0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{bmatrix}$$



# Solution (contd)

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Solving the above 15 equations gives the 15 unknowns as

$i$	$a_i$	$b_i$	$c_i$
1	0	-0.04444	7.2889
2	-1.0556	8.9278	-11.777
3	0.68943	-9.3945	36.319
4	-1.7651	28.945	-113.40
5	3.2886	-64.042	314.34



# Solution (contd)

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Therefore, the splines are given by

$$\begin{aligned}y(x) &= -0.04444 x + 7.2889, & 2.00 \leq x \leq 4.25 \\ &= -1.0556 x^2 + 8.9278 x - 11.777, & 4.25 \leq x \leq 5.25 \\ &= 0.68943 x^2 - 9.3945 x + 36.319, & 5.25 \leq x \leq 7.81 \\ &= -1.7651 x^2 + 28.945 x - 113.40, & 7.81 \leq x \leq 9.20 \\ &= 3.2886 x^2 - 64.042 x + 314.34, & 9.20 \leq x \leq 10.60\end{aligned}$$

# Solution (contd)

The length of a curve of a function  $y = f(x)$  from 'a' to 'b' is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

In this case,  $f(x)$  is defined by five separate functions a = 2.00 to b = 10.60

$$\frac{dy}{dx} = \frac{d}{dx}(-0.04444x + 7.2889), \quad 2.00 \leq x \leq 4.25$$

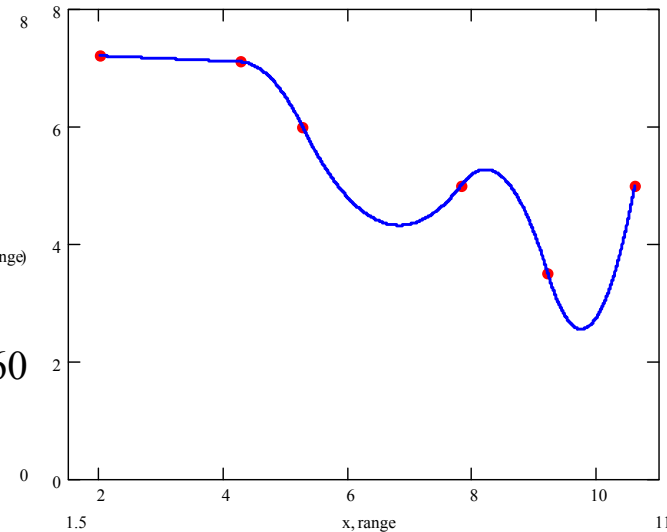
$$= \frac{d}{dx}(-1.0556x^2 + 8.9278x - 11.777), \quad 4.25 \leq x \leq 5.25$$

$$= \frac{d}{dx}(0.68943x^2 - 9.3945x + 36.319), \quad 5.25 \leq x \leq 7.81$$

$$= \frac{d}{dx}(-1.7651x^2 + 28.945x - 113.40), \quad 7.81 \leq x \leq 9.20$$

$$= \frac{d}{dx}(3.2886x^2 - 64.042x + 314.34), \quad 9.20 \leq x \leq 10.60$$

y  
f\_quadratic(range)





# Solution (contd)

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$$\begin{aligned} L &= \int_{2.00}^{4.25} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx + \int_{4.25}^{5.25} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx + \int_{5.25}^{7.81} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx + \int_{7.81}^{9.20} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx + \int_{9.20}^{10.60} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_{2.00}^{4.25} \sqrt{1 + (-0.04444)^2} dx + \int_{4.25}^{5.25} \sqrt{1 + (-2.1111x + 8.9278)^2} dx + \int_{5.25}^{7.81} \sqrt{1 + (1.3788x - 9.3945)^2} dx \\ &\quad + \int_{7.81}^{9.20} \sqrt{1 + (-3.5302x + 28.945)^2} dx + \int_{9.20}^{10.60} \sqrt{1 + (6.5772x - 64.042)^2} dx \\ &= 2.2522 + 1.5499 + 3.6596 + 2.6067 + 3.8080 \\ &= 13.876 \end{aligned}$$



# Comparison

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Compare the answer from part (a) to linear spline result and fifth order polynomial result.

We can find the length of the fifth order polynomial result in a similar fashion to the quadratic splines without breaking the integrals into five intervals. The fifth order polynomial result through the six points is given by

$$y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072923x^5, \quad 2 \leq x \leq 10.6$$

Therefore,

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_{2.00}^{10.60} \sqrt{1 + (41.344 - 31.710x + 8.3586x^2 - 0.92364x^3 + 0.036461x^4)} dx \\ &= 13.133 \end{aligned}$$